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A 4-Step Block Algorithm with Four intra points for the Direct solution of First to Fifth Order Ordinary Differential Equations

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Abstract

This paper proposes a 4-step block method for the numerical solution of first to fifth orders of ordinary differential equations (ODEs). The coefficients of the proposed scheme are obtained through interpolation and collocation techniques using a polynomial as its basis function. The derivatives were collocated at grid and off-grid points, excluding the endpoints and the method is of order 9. The stability properties of the proposed formulas are analyzed. The results obtained from our numerical experiments indicated that the new scheme outperforms other existing methods in the literature regarding accuracy when compared.

Keywords: *hybrid block, method, interpolation, collocation, power series, ODEs.*

1. INTRODUCTION

Differential Equations (DEs) are fundamental in creating models across various disciplines, including Engineering, Mathematics, Physics, Aeronautics, Elasticity, Astronomy, Dynamics, Biology, Chemistry, Medicine, Environmental Sciences, Social Sciences, and Banking. While calculus provides a means to study and find exact solutions for many equations, not all differential equations can be solved exactly. Finding exact solutions for some equations can be a daunting task. To overcome this challenge, numerical methods are often employed to solve differential equations. This paper focuses on developing a novel approach to solving initial value problems (IVPs) of Ordinary Differential Equations (ODEs) of

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various orders directly using the Linear Multistep Hybrid Block Method (LMHBMs). The ODEs are typically expressed as follows:

$$\left.\begin{array}{l} y' = f\left(x,y\right), y\left(x_0\right) = y_0 \\ y'' = f\left(x,y,y'\right), y\left(x_0\right) = y_0, y'\left(x_0\right) = y'_0 \\ y''' = f\left(x,y,y',y''\right), y\left(x_0\right) = y_0, y'\left(x_0\right) = y'_0, y''\left(x_0\right) = y''_0 \\ y^{iv} = f\left(x,y,y',y'',y'''\right), y\left(x_0\right) = y_0, y'\left(x_0\right) = y'_0, y''\left(x_0\right) = y''_0, y'''\left(x_0\right) = y'''_0 \\ y^{v} = f\left(x,y,y',y'',y''',y^{iv}\right), y\left(x_0\right) = y_0, y'\left(x_0\right) = y'_0, y''\left(x_0\right) = y''_0, y'''\left(x_0\right) = y'''_0, y^{iv}\left(x_0\right) = y^{iv}_0 \end{array}\right\} \quad (1)$$

Many authors have implemented various numerical techniques to solve these equations separately such as Badmus (2014), Omar and Kuboye (2015), Ezekiel (2018), Kuboye (2018), Abolarin (2020), Victor (2021), Roseline (2021), Raymond *et al.* (2021), Joshua (2021,2022), Faruk(2023), Morteza(2023), Muideen(2021,2023), Dauda (2022), Friday(2023), respectively derived hybrid block linear multistep method for the direct solution of general second, third and fourth IVPs of ODEs.

The blocks technique formulated by all these researchers are effective in terms of accuracy but they are constructed to solve only one of the ODEs in (1.1). However, Adeyefa and Kuboye (2020) formulated a numerical technique suitable for solving simultaneously second and third order ODEs, John (2022) developed a single numerical algorithm for solving first and second orders ODEs. Again, Olusola and Bamikole (2022) formulated a new hybrid method for direct numerical solution of non-linear second, third and fourth orders of ordinary differential equations. The aforementioned researchers did not look at the possibility of harmonizing different categories of ODEs- starting from first, second, third and fourth orders of ODEs and developing a hybrid block method capable of supplying approximate answer to the four categories of differential equations simultaneously. Therefore, this new unified computational technique with additional fifth order is formulated to solve (1). Hence, this new method rises above the setback in developing numerical methods for solving a single class of ODEs by proposing a hybrid block method that gives solution to ODEs of five different categories of order: One, Two, Three, four and five. Therefore, this proposed approach will also eliminate the burden of developing a separate method of the equations in (1).

2. MATERIALS AND METHOD

In this segment, we formulate a hybrid block linear multistep method for solving (1) using a combination of collocation and interpolation

Our approach involves representing the rough solution for the specified ODEs using power series as basis function of the form:

$$y(x) = \sum_{j=0}^{t+m-1} b_j x^j \quad (2)$$

where $b_j s$ are unknown coefficients that are to be obtained, t and m are number of interpolation and collocation points respectively. The first through the fifth derivatives of (2.1) are obtained as

$$y'(x)=\sum_{j=1}^{t+m-1} ja_j x^{j-1} \tag{3}$$

$$y''(x)=\sum_{j=2}^{t+m-1} j(j-1)a_j x^{j-2} \tag{4}$$

$$y'''(x)=\sum_{j=3}^{t+m-1} j(j-1)(j-2)a_j x^{j-3} \tag{5}$$

$$y^{iv}(x)=\sum_{j=4}^{t+m-1} j(j-1)(j-2)(j-3)a_j x^{j-4} \tag{6}$$

$$y^{v}(x)=\sum_{j=5}^{t+m-1} j(j-1)(j-2)(j-3)(j-4)a_j x^{j-5} \tag{7}$$

Interpolating (2) at $x = x_{n+2}$ and collocating equations (3) to (6) at $x = x_{n+3}$ and also collocating (7) at $x = x_n, x_{n+\frac{1}{8}}, x_{n+1}, x_{n+\frac{9}{8}}, x_{n+2}, x_{n+\frac{17}{8}}, x_{n+3}, x_{n+\frac{25}{8}}, x_{n+4}$ gives a system of nonlinear equation of the form:

$$YD=\psi \tag{8}$$

where

$$Y = (b_0, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}, b_{11}, b_{12}, b_{13})^T,$$

$$\psi = \left(y_3, \gamma_4, \delta_4, \sigma_4, \mu_4, \beta_0, \beta_{\frac{1}{8}}, \beta_1, \beta_{\frac{9}{8}}, \beta_2, \beta_{\frac{17}{8}}, \beta_{\frac{25}{8}}, \beta_3, \beta_4\right)^T \text{ and}$$

$$D=\begin{bmatrix}
1 & x_{n+3} & x_{n+3}^2 & x_{n+3}^3 & x_{n+3}^4 & x_{n+3}^5 & x_{n+3}^6 & x_{n+3}^7 & x_{n+3}^8 & x_{n+3}^9 & x_{n+3}^{10} & x_{n+3}^{11} & x_{n+3}^{12} & x_{n+3}^{13} \\
0 & 1 & 2x_{n+4} & 3x_{n+4}^2 & 4x_{n+4}^3 & 5x_{n+4}^4 & 6x_{n+4}^5 & 7x_{n+4}^6 & 8x_{n+4}^7 & 9x_{n+4}^8 & 10x_{n+4}^9 & 11x_{n+4}^{10} & 12x_{n+4}^{11} & 13x_{n+4}^{12} \\
0 & 0 & 2 & 6x_{n+4} & 12x_{n+4}^2 & 20x_{n+4}^3 & 30x_{n+4}^4 & 42x_{n+4}^5 & 56x_{n+4}^6 & 72x_{n+4}^7 & 90x_{n+4}^8 & 110x_{n+4}^9 & 132x_{n+4}^{10} & 156x_{n+4}^{11} \\
0 & 0 & 0 & 0 & 24x_{n+4} & 60x_{n+4}^2 & 120x_{n+4}^3 & 210x_{n+4}^4 & 336x_{n+4}^5 & 504x_{n+4}^6 & 720x_{n+4}^7 & 990x_{n+4}^8 & 1320x_{n+4}^9 & 1716x_{n+4}^{10} \\
0 & 0 & 0 & 0 & 24 & 120x_{n+4} & 360x_{n+4}^2 & 480x_{n+4}^3 & 1680x_{n+4}^4 & 3024x_{n+4}^5 & 5040x_{n+4}^6 & 7920x_{n+4}^7 & 11880x_{n+4}^8 & 17160x_{n+4}^9 \\
0 & 0 & 0 & 0 & 0 & 120 & 720x_n & 1440x_n^2 & 6720x_n^3 & 15120x_n^4 & 30240x_n^5 & 55440x_n^6 & 95040x_n^7 & 154440x_n^8 \\
0 & 0 & 0 & 0 & 0 & 120 & 720x_{n+\frac{1}{8}} & 1440x_{n+\frac{1}{8}}^2 & 6720x_{n+\frac{1}{8}}^3 & 15120x_{n+\frac{1}{8}}^4 & 30240x_{n+\frac{1}{8}}^5 & 55440x_{n+\frac{1}{8}}^6 & 95040x_{n+\frac{1}{8}}^7 & 154440x_{n+\frac{1}{8}}^8 \\
0 & 0 & 0 & 0 & 0 & 120 & 720x_{n+1} & 1440x_{n+1}^2 & 6720x_{n+1}^3 & 15120x_{n+1}^4 & 30240x_{n+1}^5 & 55440x_{n+1}^6 & 95040x_{n+1}^7 & 154440x_{n+1}^8 \\
0 & 0 & 0 & 0 & 0 & 120 & 720x_{n+\frac{9}{8}} & 1440x_{n+\frac{9}{8}}^2 & 6720x_{n+\frac{9}{8}}^3 & 15120x_{n+\frac{9}{8}}^4 & 30240x_{n+\frac{9}{8}}^5 & 55440x_{n+\frac{9}{8}}^6 & 95040x_{n+\frac{9}{8}}^7 & 154440x_{n+\frac{9}{8}}^8 \\
0 & 0 & 0 & 0 & 0 & 120 & 720x_{n+2} & 1440x_{n+2}^2 & 6720x_{n+2}^3 & 15120x_{n+2}^4 & 30240x_{n+2}^5 & 55440x_{n+2}^6 & 95040x_{n+2}^7 & 154440x_{n+2}^8 \\
0 & 0 & 0 & 0 & 0 & 120 & 720x_{n+\frac{17}{8}} & 1440x_{n+\frac{17}{8}}^2 & 6720x_{n+\frac{17}{8}}^3 & 15120x_{n+\frac{17}{8}}^4 & 30240x_{n+\frac{17}{8}}^5 & 55440x_{n+\frac{17}{8}}^6 & 95040x_{n+\frac{17}{8}}^7 & 154440x_{n+\frac{17}{8}}^8 \\
0 & 0 & 0 & 0 & 0 & 120 & 720x_{n+3} & 1440x_{n+3}^2 & 6720x_{n+3}^3 & 15120x_{n+3}^4 & 30240x_{n+3}^5 & 55440x_{n+3}^6 & 95040x_{n+3}^7 & 154440x_{n+3}^8 \\
0 & 0 & 0 & 0 & 0 & 120 & 720x_{n+\frac{25}{8}} & 1440x_{n+\frac{25}{8}}^2 & 6720x_{n+\frac{25}{8}}^3 & 15120x_{n+\frac{25}{8}}^4 & 30240x_{n+\frac{25}{8}}^5 & 55440x_{n+\frac{25}{8}}^6 & 95040x_{n+\frac{25}{8}}^7 & 154440x_{n+\frac{25}{8}}^8 \\
0 & 0 & 0 & 0 & 0 & 120 & 720x_n & 1440x_{n+4}^2 & 6720x_{n+4}^3 & 15120x_{n+4}^4 & 30240x_{n+4}^5 & 55440x_{n+4}^6 & 95040x_{n+4}^7 & 154440x_{n+4}^8
\end{bmatrix}$$

Equation (8) is solved by matrix inversion technique to obtain $b_j s$ which are then inserted into equation (2) to derive a continuous linear multi-step method (LMM) of the form:

$$y(x)=\alpha_3(x)y_{n+3}+h\gamma_4(x)y'_{n+4}+h^2\delta_4(x)y''_{n+4}+h^3\sigma_4(x)y'''_{n+4}+h^4\mu_4(x)y^{iv}_{n+4}$$
$$+h^5\begin{pmatrix}\beta_0(x)f_n+\beta_{\frac{1}{8}}(x)f_{n+\frac{1}{8}}+\beta_1(x)f_{n+1}+\beta_{\frac{9}{8}}(x)f_{n+\frac{9}{8}}+\beta_2(x)f_{n+2}\\ +\beta_{\frac{17}{8}}(x)f_{n+\frac{17}{8}}+\beta_3(x)f_{n+3}+\beta_{\frac{25}{8}}(x)f_{n+\frac{25}{8}}+\beta_4(x)f_{n+4}\end{pmatrix} \quad (9)$$

where

$$\left.\begin{array}{l}\alpha_3(x)=1\\ \gamma_4(x)=-3h+x\\ \delta_4(x)=\frac{1}{2}x^2+\frac{15}{2}h^2-4hx\\ \sigma_4(x)=\frac{1}{6}x^3-2hx^2-\frac{21}{2}h^3+8h^2x\\ \mu_4(x)=-\frac{2}{3}hx^3+\frac{85}{8}h^4+\frac{1}{24}x^4+4h^2x^2-\frac{32}{3}h^3x\end{array}\right\} \quad (10)$$

$$\beta_0(x)=\left.\begin{array}{l}\frac{8639924959}{6301152000}h^5+\frac{3296819}{231336000}\frac{x^7}{h^2}-\frac{5387936}{3614625}h^4x-\frac{261437}{34272000}\frac{x^8}{h^3}+\frac{2729396}{4417875}h^3x^2\\ +\frac{1238843}{462672000}\frac{x^9}{h^4}-\frac{26368}{212625}h^2x^3-\frac{1991}{3213000}\frac{x^{10}}{h^5}+\frac{556957}{43375500}hx^4\\ +\frac{403}{4417875}\frac{x^{11}}{h^6}+\frac{1}{120}x^5-\frac{4}{516375}\frac{x^{12}}{h^7}-\frac{179971}{11016000}\frac{x^6}{h}+\frac{64}{221524875}\frac{x^{13}}{h^8}\end{array}\right\} \quad (11)$$

$$\beta_{\frac{1}{8}}(x)=\left.\begin{array}{l}\frac{4192}{333523575}\frac{x^{12}}{h^7}+\frac{272}{14973}\frac{x^6}{h}-\frac{5894242304}{2334665025}h^4x-\frac{9390681413}{4046752710}h^5-\frac{116903936}{111174525}h^3x^2\\ -\frac{113968}{778221675}\frac{x^{11}}{h^6}-\frac{26368}{212625}h^2x^3-\frac{125308}{30320325}\frac{x^9}{h^4}-\frac{2048}{4335806475}\frac{x^{13}}{h^8}\\ +\frac{46925824}{212242275}h^2x^3-\frac{460568}{23582475}\frac{x^7}{h^2}-\frac{7786496}{212242275}hx^4+\frac{41581}{42448455}\frac{x^{10}}{h^5}+\frac{533411}{47164950}\frac{x^8}{h^3}\end{array}\right\} \quad (12)$$

$$\beta_1(x)=\left.\begin{array}{l}\frac{497970816313}{18525386880}h^5+\frac{164671}{2698920}\frac{x^7}{h^2}-\frac{2308793}{43182720}\frac{x^8}{h^3}-\frac{975782912}{33399135}h^4x+\frac{342439}{13880160}\frac{x^9}{h^4}\\ -\frac{3779648}{318087}h^3x^2-\frac{41233}{6072570}\frac{x^{10}}{h^5}-\frac{6187168}{3036285}h^2x^3-\frac{12524}{11133045}\frac{x^{11}}{h^6}+\frac{202024}{3036285}hx^4\\ -\frac{496}{4771305}\frac{x^{12}}{h^6}-\frac{5}{252}\frac{x^5}{h}-\frac{256}{62026965}\frac{x^{13}}{h^8}\end{array}\right\} \quad (13)$$

$$
\left.\begin{aligned}
&-\frac{21802134811}{652702050}h^{5}+\frac{272}{13041}\frac{x^{6}}{h}-\frac{189932}{6846525}\frac{x^{9}}{h^{4}}+\frac{1312}{10758825}\frac{x^{12}}{h^{7}}-\frac{2871296}{20539575}hx^{4}\\
\beta_{\frac{9}{8}}(x)=&-\frac{2745761792}{75311775}h^{4}x+\frac{169}{21735}\frac{x^{10}}{h^{5}}+\frac{2664448}{978075}h^{2}x^{3}-\frac{2048}{419594175}\frac{x^{13}}{h^{8}}+\frac{25161728}{1673595}h^{3}x^{2}\\
&-\frac{443656}{6846525}\frac{x^{7}}{h^{2}}-\frac{10928}{8367975}\frac{x^{11}}{h^{6}}+\frac{29747}{507150}\frac{x^{8}}{h^{3}}
\end{aligned}\right\} \tag{14}
$$

$$
\left.\begin{aligned}
&\frac{39574742173}{544864320}h^{5}-\frac{720232}{127575}h^{2}x^{3}+\frac{5398}{3274425}\frac{x^{11}}{h^{6}}+\frac{128}{18243225}\frac{x^{13}}{h^{8}}-\frac{294011}{893025}hx^{4}\\
\beta_{2}(x)=&-\frac{773951744}{9823275}h^{4}x-\frac{232}{1403325}\frac{x^{12}}{h^{7}}+\frac{1657969}{57153600}\frac{x^{9}}{h^{4}}-\frac{17}{1008}\frac{x^{6}}{h}+\frac{104586992}{3274425}h^{3}x^{2}\\
&+\frac{172321}{3175200}\frac{x^{7}}{h^{2}}-\frac{919}{102060}\frac{x^{10}}{h^{5}}-\frac{173741}{3175200}\frac{x^{8}}{h^{3}}
\end{aligned}\right\} \tag{15}
$$

$$
\left.\begin{aligned}
&-\frac{4319918129}{57891834}h^{5}-\frac{1867231232}{55665225}h^{3}x^{2}-\frac{398116}{15181425}\frac{x^{9}}{h^{4}}-\frac{118784}{309825}hx^{4}+\frac{278069248}{3408075}h^{4}x\\
\beta_{\frac{17}{8}}(x)=&+\frac{16}{1071}\frac{x^{6}}{h}+\frac{24989}{3036285}\frac{x^{10}}{h^{5}}+\frac{4721}{96390}\frac{x^{8}}{h^{3}}+\frac{5441536}{893025}h^{2}x^{3}-\frac{81304}{1686825}\frac{x^{7}}{h^{2}}\\
&-\frac{2048}{310134825}\frac{x^{13}}{h^{8}}-\frac{12112}{7952175}\frac{x^{11}}{h^{6}}-\frac{736}{4771305}\frac{x^{12}}{h^{7}}
\end{aligned}\right\} \tag{16}
$$

$$
\left.\begin{aligned}
&\frac{613414812533}{13924310400}h^{5}-\frac{191}{101430}\frac{x^{10}}{h^{5}}-\frac{256}{139864725}\frac{x^{13}}{h^{8}}-\frac{17}{5796}\frac{x^{6}}{h}-\frac{24196096}{512325}h^{4}x\\
\beta_{3}(x)=&+\frac{2532064}{760725}h^{2}x^{3}+\frac{174871}{18257400}\frac{x^{7}}{h^{2}}+\frac{413723}{73029600}\frac{x^{9}}{h^{4}}+\frac{148}{398475}\frac{x^{11}}{h^{6}}-\frac{10539328}{557865}h^{3}x^{2}\\
&-\frac{15593}{1545600}\frac{x^{8}}{h^{3}}-\frac{16}{398475}\frac{x^{12}}{h^{7}}-\frac{29608}{139725}hx^{4}
\end{aligned}\right\} \tag{17}
$$

$$
\left.\begin{aligned}
&-\frac{42760483063}{1033782750}h^{5}-\frac{5099368448}{278326125}h^{3}x^{2}+\frac{100553}{75907125}\frac{x^{10}}{h^{5}}+\frac{250955776}{75907125}h^{2}x^{3}\\
\beta_{\frac{25}{9}}(x)=&-\frac{16633856}{75907125}hx^{4}-\frac{2048}{1550674125}\frac{x^{13}}{h^{8}}+\frac{3424}{119282625}\frac{x^{12}}{h^{7}}-\frac{56024}{8434125}\frac{x^{7}}{h^{2}}\\
&-\frac{300676}{75907125}\frac{x^{9}}{h^{4}}-\frac{73264}{278326125}\frac{x^{11}}{h^{6}}+\frac{16}{7875}\frac{x^{6}}{h}+\frac{37560713216}{834978375}h^{4}x+\frac{118427}{16868250}\frac{x^{8}}{h^{3}}
\end{aligned}\right\} \tag{18}
$$

$$
\left.\begin{aligned}
&-\frac{18556031051}{10571109120}h^{5}-\frac{4}{13340943}\frac{x^{12}}{h^{7}}+\frac{986929}{27167011200}\frac{x^{9}}{h^{4}}+\frac{64}{4335806475}\frac{x^{13}}{h^{8}}-\frac{17}{958272}\frac{x^{6}}{h}\\
\beta_4(x)=&-\frac{556289204}{778221675}h^{3}x^{2}+\frac{26402444}{212242275}h^{2}x^{3}-\frac{15163}{241484544}\frac{x^{8}}{h^{3}}+\frac{88073}{1509278400}\frac{x^{7}}{h^{2}}\\
&+\frac{4272511744}{2334665025}h^{4}x-\frac{6936803}{848969100}hx^{4}-\frac{853}{67917528}\frac{x^{10}}{h^{5}}+\frac{2027}{778221675}\frac{x^{11}}{h^{6}}
\end{aligned}\right\} \quad (19)
$$

Equations (10) to (19) are then inserted into equation (9) to obtain the required continuous scheme and then evaluating same, that is (9) at $x = x_{n+j},\ j = 0, \frac{1}{8}, 1, \frac{9}{8}, 2, \frac{17}{8}, \frac{25}{8}, 4$ gives the 4-step block method with 4 off-grid points as follows:

$$
\left.\begin{aligned}
&y_{n}\\
&=y_{n+3}-3hz_{n+4}+\frac{15}{2}h^{2}g_{n+4}-\frac{21}{2}h^{3}p_{n+4}+\frac{85}{8}h^{4}q_{n+4}+\frac{8639924959}{6301152000}h^{5}f_{n}-\frac{9390681413}{4046752710}h^{5}f_{n+\frac{1}{8}}\\
&+\frac{497970816313}{18525386880}h^{5}f_{n+1}-\frac{21802134811}{652702050}h^{5}f_{n+\frac{9}{8}}+\frac{39574742173}{544864320}h^{5}f_{n+2}-\frac{4319918129}{57891834}h^{5}f_{n+\frac{17}{8}}\\
&+\frac{613414812533}{13924310400}h^{5}f_{n+3}-\frac{42760483063}{1033782750}h^{5}f_{n+\frac{25}{8}}+\frac{18556031051}{10571109120}h^{5}f_{n+4}
\end{aligned}\right\} \quad (20)
$$

$$
\left.\begin{aligned}
&y_{n+\frac{1}{8}}\\
&=y_{n+3}-\frac{23}{8}hz_{n+4}+\frac{897}{128}h^{2}g_{n+4}-\frac{29279}{3072}h^{3}p_{n+4}+\frac{306475}{32768}h^{4}q_{n+4}-\frac{950183708594237 41}{296004206827929600}h^{5}f_{n}\\
&-\frac{2359620619628065907}{4345167636112343040}h^{5}f_{n+\frac{1}{8}}+\frac{389821286453580689}{62160883433865216}h^{5}f_{n+1}-\frac{217940870507974885}{28033339587821568}h^{5}f_{n+\frac{9}{8}}\\
&+\frac{618654271072557739}{36565225549332480}h^{5}f_{n+2}-\frac{535530413013051376 3}{310804417169326080}h^{5}f_{n+\frac{17}{8}}+\frac{97501671439181675}{9344446529273856}h^{5}f_{n+3}\\
&-\frac{14901501993370567259}{1554022085846630400}h^{5}f_{n+\frac{25}{8}}-\frac{7599200876323478689}{17380670544449372160}h^{5}f_{n+4}
\end{aligned}\right\} \quad (21)
$$

$$
\left.\begin{aligned}
&y_{n+1}\\
&=y_{n+3}-2hz_{n+4}+4h^{2}g_{n+4}-\frac{13}{3}h^{3}p_{n+4}+\frac{10}{3}h^{4}q_{n+4}+\frac{12834431081}{33081048000}h^{5}f_{n}-\frac{2845820288}{4335806475}h^{5}f_{n+\frac{1}{8}}\\
&+\frac{7525093327}{992431440}h^{5}f_{n+1}-\frac{1840793984}{195810615}h^{5}f_{n+\frac{9}{8}}+\frac{7605747377}{371498400}h^{5}f_{n+2}-\frac{588320896}{28194075}h^{5}f_{n+\frac{17}{8}}\\
&+\frac{1880055553}{149189040}h^{5}f_{n+3}-\frac{126068153984}{10854718875}h^{5}f_{n+\frac{25}{8}}-\frac{139699821329}{1553953040640}h^{5}f_{n+4}
\end{aligned}\right\} \quad (22)
$$

$$\left.\begin{aligned}&y_{n+\frac{9}{8}}\\&=y_{n+3}-\frac{15}{8}hz_{n+4}+\frac{465}{128}h^2g_{n+4}-\frac{3885}{1024}h^3p_{n+4}+\frac{91915}{32768}h^4q_{n+4}\\&+\frac{95018370859423741}{296004206827929600}h^5f_n-\frac{2359620619628065907}{4345167636112343040}h^5f_{n+\frac{1}{8}}+\frac{389821286453580689}{62160883433865216}h^5f_{n+1}\\&-\frac{217940870507974885}{28033339587821568}h^5f_{n+\frac{9}{8}}+\frac{618654271072557739}{36565225549332480}h^5f_{n+2}-\frac{5355304130130513763}{310804417169326080}h^5f_{n+\frac{17}{8}}\\&+\frac{97501671439181675}{9344446529273856}h^5f_{n+3}-\frac{14901501993370567259}{1554022085846630400}h^5f_{n+\frac{25}{8}}-\frac{7599200876323478689}{17380670544449372160}h^5f_{n+4}\end{aligned}\right\}\qquad(23)$$

$$\left.\begin{aligned}&y_{n+2}\\&=y_{n+3}-hz_{n+4}+\frac{3}{2}h^2g_{n+4}-\frac{7}{6}h^3p_{n+4}+\frac{5}{8}h^4q_{n+4}+\frac{965330953}{15878903040}h^5f_n-\frac{6241290763}{60701290650}h^5f_{n+\frac{1}{8}}\\&+\frac{65824381099}{55576160640}h^5f_{n+1}-\frac{8620466419}{5874318450}h^5f_{n+\frac{9}{8}}+\frac{25913788211}{8172964800}h^5f_{n+2}-\frac{558633875}{173675502}h^5f_{n+\frac{17}{8}}\\&+\frac{246637751837}{125318793600}h^5f_{n+3}-\frac{1533154747}{868377510}h^5f_{n+\frac{25}{8}}-\frac{139699821329}{1553953040640}h^5f_{n+4}\end{aligned}\right\}\qquad(24)$$

$$\left.\begin{aligned}&y_{n+\frac{17}{8}}\\&=y_{n+3}-\frac{7}{8}hz_{n+4}+\frac{161}{128}h^2g_{n+4}-\frac{2863}{3072}h^3p_{n+4}+\frac{46529}{98304}h^4q_{n+4}+\frac{5656711783152797}{126858945783398400}h^5f_n\\&-\frac{12766855711660573}{169292245562818560}h^5f_{n+\frac{1}{8}}+\frac{115667532756976349}{133201893072568320}h^5f_{n+1}-\frac{323131558629626321}{300357209869516800}h^5f_{n+\frac{9}{8}}\\&+\frac{36388326867793819}{15670810949713920}h^5f_{n+2}-\frac{313675764286222319}{133201893072568320}h^5f_{n+\frac{17}{8}}+\frac{13102708483958417}{9101733632409600}h^5f_{n+3}\\&-\frac{122376733552909889}{95144209337548800}h^5f_{n+\frac{25}{8}}-\frac{71314165792874519}{1064122686394859520}h^5f_{n+4}\end{aligned}\right\}\qquad(25)$$

$$\left.\begin{aligned}&y_{n+\frac{25}{8}}\\&=y_{n+3}+\frac{1}{8}hz_{n+4}-\frac{15}{128}h^2g_{n+4}+\frac{169}{3072}h^3p_{n+4}-\frac{565}{3276}h^4q_{n+4}-\frac{14930495911683631}{133201893072568320000}h^5f_n\\&+\frac{24701769674560999}{1303550290837029120}h^5f_{n+\frac{1}{8}}-\frac{20269061770100201}{932413251507978240}h^5f_{n+1}+\frac{169743955818152699}{6307501407259852800}h^5f_{n+\frac{9}{8}}\\&-\frac{6315987646550279}{109695676647997440}h^5f_{n+2}+\frac{10865200810401643}{186482650301595648}h^5f_{n+\frac{17}{8}}-\frac{72352571046993373}{2102500469086617600}h^5f_{n+3}\\&-\frac{694600050329202923}{23310331287699456000}h^5f_{n+\frac{25}{8}}+\frac{106371638314375061}{52142011633348116480}h^5f_{n+4}\end{aligned}\right\}\qquad(26)$$

$$\left.\begin{aligned}&y_{n+4}\\&= y_{n+3} + hz_{n+4} - \frac{1}{2}h^2 g_{n+4} + \frac{1}{6}h^3 p_{n+4} - \frac{1}{24}h^4 q_{n+4} - \frac{60320753}{26464838400}h^5 f_n + \frac{4252211}{1103659830}h^5 f_{n+\frac{1}{8}}\\&- \frac{2453409877}{55576160640}h^5 f_{n+1} + \frac{106985647}{1958106150}h^5 f_{n+\frac{9}{8}} - \frac{190566953}{1634592960}h^5 f_{n+2} + \frac{102380833}{868377510}h^5 f_{n+\frac{17}{8}}\\&- \frac{2875280801}{41772931200}h^5 f_{n+3} + \frac{257372761}{4341887550}h^5 f_{n+\frac{25}{8}} + \frac{7003235567}{1553953040640}h^5 f_{n+4}\end{aligned}\right\} \quad (27)$$

Obtaining the first derivative of (9) and evaluating at $x = x_n, x_{n+\frac{1}{8}}, x_{n+1}, x_{n+\frac{9}{8}}, x_{n+2}, x_{n+\frac{17}{8}}, x_{n+3}, x_{n+\frac{25}{8}}$ gives the following discrete schemes:

$$\left.\begin{aligned}&z_n\\&= z_{n+4} - 4hg_{n+4} + 8h^2 p_{n+4} - \frac{32}{3}h^3 q_{n+4} - \frac{343724077613507849}{256157486678016000}h^4 f_n + \frac{5894242304}{2334665025}h^4 f_{n+\frac{1}{8}}\\&- \frac{975782912}{33399135}h^4 f_{n+1} + \frac{2745761792}{75311775}h^4 f_{n+\frac{9}{8}} - \frac{773951744}{9823275}h^4 f_{n+2} + \frac{278069248}{3408075}h^4 f_{n+\frac{17}{8}}\\&- \frac{24196096}{512325}h^4 f_{n+3} + \frac{37560713216}{834978375}h^4 f_{n+\frac{25}{8}} + \frac{4272511744}{2334665025}h^4 f_{n+4}\end{aligned}\right\} \quad (28)$$

$$\left.\begin{aligned}&z_{n+\frac{1}{8}}\\&= z_{n+4} - \frac{31}{8}hg_{n+4} + \frac{961}{128}h^2 p_{n+4} - \frac{29791}{3072}h^3 q_{n+4} - \frac{5387936}{3614625}h^4 f_n + \frac{3827391885671593}{1684695888691200}h^4 f_{n+\frac{1}{8}}\\&- \frac{472306881822059041}{17931024067461120}h^4 f_{n+1} + \frac{497725090196326037}{15162262998220800}h^4 f_{n+\frac{9}{8}} - \frac{249854653956772243}{3515887072051200}h^4 f_{n+2}\\&+ \frac{278069823571271472177861248}{11206890042163200}h^4 f_{n+\frac{17}{8}} - \frac{1724831752979724497}{40432701328588800}h^4 f_{n+3}\\&+ \frac{9838030666602179}{242573377536000}h^4 f_{n+\frac{25}{8}} + \frac{38286165444281327}{23104400759193600}h^4 f_{n+4}\end{aligned}\right\} \quad (29)$$

$$\left.\begin{aligned}&z_{n+1}\\&= z_{n+4} - 3hg_{n+4} + \frac{9}{2}h^2 p_{n+4} - \frac{9}{2}h^3 q_{n+4} - \frac{2406303}{4188800}h^4 f_n + \frac{9338922}{9607675}h^4 f_{n+\frac{1}{8}}\\&- \frac{3299379}{293216}h^4 f_{n+1} + \frac{4326174}{309925}h^4 f_{n+\frac{9}{8}} - \frac{26301411}{862400}h^4 f_{n+2}\\&+ \frac{7152942}{229075}h^4 f_{n+\frac{17}{8}} - \frac{16748883}{901600}h^4 f_{n+3}\\&+ \frac{568746}{32725}h^4 f_{n+\frac{25}{8}} + \frac{130168233}{175683200}h^4 f_{n+4}\end{aligned}\right\} \quad (30)$$

$$\left.\begin{aligned}
&z_{n+\frac{9}{8}}\\
&=z_{n+4}-\frac{23}{8}hg_{n+4}+\frac{529}{128}h^{2}p_{n+4}-\frac{12167}{3072}h^{3}q_{n+4}-\frac{425408085857724579}{85385828892672000}h^{4}f_{n}\\
&+\frac{5742430492221427}{6812031202099200}h^{4}f_{n+\frac{1}{8}}-\frac{174882713967366721}{17931024067461120}h^{4}f_{n+1}+\frac{106324627815529}{8789717680128}h^{4}f_{n+\frac{9}{8}}\\
&-\frac{25343397644783851}{958878292377600}h^{4}f_{n+2}+\frac{302864366425469749}{11206890042163200}h^{4}f_{n+\frac{17}{8}}-\frac{378385518556357}{23439247147008}h^{4}f_{n+3}\\
&+\frac{120551489446586291}{8004921458688000}h^{4}f_{n+\frac{25}{8}}+\frac{20164212168488791}{31140714066739200}h^{4}f_{n+4}
\end{aligned}\right\}\qquad(31)$$

$$\left.\begin{aligned}
&z_{n+2}\\
&=z_{n+4}-2hg_{n+4}+2h^{2}p_{n+4}-\frac{4}{3}h^{3}q_{n+4}-\frac{17244688}{119282625}h^{4}f_{n}+\frac{190308352}{778221675}h^{4}f_{n+\frac{1}{8}}\\
&-\frac{94203608}{33399135}h^{4}f_{n+1}+\frac{789782528}{789782528}h^{4}f_{n+\frac{9}{8}}-\frac{24817762}{3274425}h^{4}f_{n+2}+\frac{1285464064}{166995675}h^{4}f_{n+\frac{17}{8}}\\
&-\frac{355281496}{75311775}h^{4}f_{n+3}+\frac{170500096}{39760875}h^{4}f_{n+\frac{25}{8}}+\frac{67586584}{333523575}h^{4}f_{n+4}
\end{aligned}\right\}\qquad(32)$$

$$\left.\begin{aligned}
&z_{n+\frac{17}{8}}\\
&=z_{n+4}-\frac{15}{8}hg_{n+4}+\frac{225}{128}h^{2}p_{n+4}-\frac{1125}{1024}h^{3}q_{n+4}-\frac{324440389125}{2811056095232}h^{4}f_{n}+\frac{5034779589375}{25790406197248}h^{4}f_{n+\frac{1}{8}}\\
&-\frac{11071557403125}{4919348166656}h^{4}f_{n+1}+\frac{2320310773125}{831948587008}h^{4}f_{n+\frac{9}{8}}-\frac{3494877575625}{578746843136}h^{4}f_{n+2}+\frac{3769284019125}{614918520832}h^{4}f_{n+\frac{17}{8}}\\
&-\frac{25040910931875}{6655588696064}h^{4}f_{n+3}+\frac{299005932375}{87845502976}h^{4}f_{n+\frac{25}{8}}+\frac{2768527081125}{16842714251264}h^{4}f_{n+4}
\end{aligned}\right\}\qquad(33)$$

$$\left.\begin{aligned}
&z_{n+3}\\
&=z_{n+4}-hg_{n+4}+\frac{1}{2}h^{2}p_{n+4}-\frac{1}{6}h^{3}q_{n+4}-\frac{58258289}{5089392000}h^{4}f_{n}+\frac{45183074}{2334665025}h^{4}f_{n+\frac{1}{8}}-\frac{237401251}{1068772320}h^{4}f_{n+1}\\
&+\frac{828470}{3012471}h^{4}f_{n+\frac{9}{8}}-\frac{370346411}{628689600}h^{4}f_{n+2}+\frac{9052166}{15181425}h^{4}f_{n+\frac{17}{8}}-\frac{11378069}{32133024}h^{4}f_{n+3}\\
&+\frac{5240402}{17040375}h^{4}f_{n+\frac{25}{8}}+\frac{864375803}{42691017600}h^{4}f_{n+4}
\end{aligned}\right\}\qquad(34)$$

$$\left.\begin{aligned}
&z_{n+\frac{25}{8}}\\
&=z_{n+4}-\frac{7}{8}hg_{n+4}+\frac{49}{128}h^{2}p_{n+4}-\frac{343}{3072}h^{3}q_{n+4}-\frac{247788999392207}{36593926668288000}h^{4}f_{n}+\frac{12200212880023}{1065828011212800}h^{4}f_{n+\frac{1}{8}}\\
&-\frac{48021131545681}{365939266682880}h^{4}f_{n+1}+\frac{4569345863551}{28130358067200}h^{4}f_{n+\frac{9}{8}}-\frac{24899684602723}{71752797388800}h^{4}f_{n+2}\\
&+\frac{80285821685797}{228712041676800}h^{4}f_{n+\frac{17}{8}}-\frac{169919996625281}{825157169971200}h^{4}f_{n+3}+\frac{67725103440967}{381186736128000}h^{4}f_{n+\frac{25}{8}}\\
&+\frac{1307889116580263}{102319489076428800}h^{4}f_{n+4}
\end{aligned}\right\}\qquad(35)$$

Obtaining the second derivative of (2.8) and evaluating at $x = x_n, x_{n+\frac{1}{8}}, x_{n+1}, x_{n+\frac{9}{8}}\ , x_{n+2}\ , x_{n+\frac{17}{8}}\ , x_{n+3}, x_{n+\frac{25}{8}}$ gives the following discrete schemes:

$$\left.\begin{aligned}
g_{n} &= g_{n+4} - 4h p_{n+4} + 8h^2 q_{n+4} + \frac{5458792}{4417875} h^3 f_{n} - \frac{233807872}{111174525} h^3 f_{n+\frac{1}{8}} \\
&+ \frac{7559296}{318087} h^3 f_{n+1} - \frac{50323456}{1673595} h^3 f_{n+\frac{9}{8}} + \frac{209173984}{3274425} h^3 f_{n+2} - \frac{3734462464}{55665225} h^4 f_{n+\frac{17}{8}} \\
&+ \frac{21078656}{557865} h^3 f_{n+3} - \frac{10198736896}{278326125} h^4 f_{n+\frac{25}{8}} - \frac{1112578408}{778221675} h^3 f_{n+4}
\end{aligned}\right\} \quad (36)$$

$$\left.\begin{aligned}
g_{n+\frac{1}{8}} &= g_{n+4} - \frac{31}{8} h p_{n+4} + \frac{961}{128} h^2 q_{n+4} + \frac{122233677539503}{106732286115840} h^3 f_{n} - \frac{9747256248241}{5013975859200} h^3 f_{n+\frac{1}{8}} \\
&+ \frac{593670594156127}{26683071528960} h^3 f_{n+1} - \frac{8861102801804243}{315880479129600} h^3 f_{n+\frac{9}{8}} + \frac{2186758425015881}{36623823667200} h^3 f_{n+2} \\
&- \frac{14613244565772491}{233476875878400} h^3 f_{n+\frac{17}{8}} + \frac{14879177256655069}{421173972172800} h^3 f_{n+3} - \frac{532399344060457}{15565125058560} h^3 f_{n+\frac{25}{8}} \\
&- \frac{2253913981991807}{1684695888691200} h^3 f_{n+4}
\end{aligned}\right\} \quad (37)$$

$$\left.\begin{aligned}
g_{n+1} &= g_{n+4} - 3h p_{n+4} + \frac{9}{2} h^2 q_{n+4} + \frac{669807}{1047200} h^3 f_{n} - \frac{212274}{196075} h^3 f_{n+\frac{1}{8}} \\
&+ \frac{263601}{20944} h^3 f_{n+1} - \frac{4846806}{309925} h^3 f_{n+\frac{9}{8}} + \frac{7353387}{215600} h^3 f_{n+2} - \frac{8076582}{229075} h^4 f_{n+\frac{17}{8}} \\
&+ \frac{21078656}{4958800} h^3 f_{n+3} - \frac{4456062}{229075} h^4 f_{n+\frac{25}{8}} - \frac{240167079}{307445600} h^3 f_{n+4}
\end{aligned}\right\} \quad (38)$$

$$\left.\begin{aligned}
g_{n+\frac{9}{8}} &= g_{n+4} - \frac{23}{8} h p_{n+4} + \frac{529}{128} h^2 q_{n+4} + \frac{24597688370641}{42354081792000} h^3 f_{n} - \frac{139494787121453}{141917316710400} h^3 f_{n+\frac{1}{8}} \\
&+ \frac{125401014917953}{10987147100160} h^3 f_{n+1} - \frac{4325333252201}{305198530560} h^3 f_{n+\frac{9}{8}} + \frac{3405978253958387}{109871471001600} h^3 f_{n+2} \\
&- \frac{1066556995032877}{33353839411200} h^4 f_{n+\frac{17}{8}} + \frac{1079606709289}{58133053440} h^3 f_{n+3} - \frac{1214063314603523}{68669669376000} h^4 f_{n+\frac{25}{8}} \\
&- \frac{1620911220278111}{2270677067366400} h^3 f_{n+4}
\end{aligned}\right\} \quad (39)$$

$$\left.\begin{aligned}
&g_{n+\frac{17}{8}}\\
&=g_{n+4}-\frac{15}{8}hp_{n+4}+\frac{225}{128}h^2q_{n+4}+\frac{2706405975}{12549357568}h^3f_n-\frac{588100543875}{588100543875}h^3f_{n+\frac{1}{8}}\\
&+\frac{647905224375}{153729630208}h^3f_{n+1}-\frac{38811783375}{7428112384}h^3f_{n+\frac{9}{8}}+\frac{4200098625}{369098752}h^3f_{n+2}\\
&-\frac{444833780625}{38432407552}h^4f_{n+\frac{17}{8}}+\frac{1471399165125}{207987146752}h^3f_{n+3}-\frac{251097135525}{38432407552}h^4f_{n+\frac{25}{8}}\\
&-\frac{7348283951625}{25790406197248}h^3f_{n+4}
\end{aligned}\right\}\quad(40)$$

$$\left.\begin{aligned}
&g_{n+3}\\
&=g_{n+4}-hp_{n+4}+\frac{1}{2}h^2q_{n+4}+\frac{897593}{20196000}h^3f_n-\frac{58488946}{778221675}h^3f_{n+\frac{1}{8}}+\frac{153988481}{178128720}h^3f_{n+1}\\
&-\frac{256006}{239085}h^3f_{n+\frac{9}{8}}+\frac{17250313}{7484400}h^3f_{n+2}-\frac{130017106}{55665225}h^4f_{n+\frac{17}{8}}+\frac{1407451}{991760}h^3f_{n+3}\\
&-\frac{347527234}{278326125}h^4f_{n+\frac{25}{8}}-\frac{1725996449}{24903093600}h^3f_{n+4}
\end{aligned}\right\}\quad(41)$$

$$\left.\begin{aligned}
&g_{n+\frac{25}{8}}\\
&=g_{n+4}-\frac{7}{8}hp_{n+4}+\frac{49}{128}h^2q_{n+4}+\frac{11744381728681}{381186736128000}h^3f_n-\frac{1156768553353}{22204750233600}h^3f_{n+\frac{1}{8}}\\
&+\frac{456335819933}{762373472256}h^3f_{n+1}-\frac{955703161399}{1289308078080}h^3f_{n+\frac{9}{8}}+\frac{1189594967729}{747424972800}h^3f_{n+2}\\
&-\frac{7680593039819}{4764834201600}h^4f_{n+\frac{17}{8}}+\frac{1662765684881}{1719077437440}h^3f_{n+3}-\frac{6702181816897}{7941390336000}h^4f_{n+\frac{25}{8}}\\
&-\frac{53966586675113}{1065828011212800}h^3f_{n+4}
\end{aligned}\right\}\quad(42)$$

Obtaining the third derivative of (9) and evaluating at $x=x_n, x_{n+\frac{1}{8}}, x_{n+1}, x_{n+\frac{9}{8}}, x_{n+2}, x_{n+\frac{17}{8}}, x_{n+3}, x_{n+\frac{25}{8}}$ gives the following discrete schemes:

$$\left.\begin{aligned}
&p_n\\
&=p_{n+4}-4hq_{n+4}-\frac{52736}{70875}h^2f_n+\frac{93851648}{70747425}h^2f_{n+\frac{1}{8}}-\frac{12374336}{1012095}h^2f_{n+1}\\
&+\frac{5328896}{326025}h^2f_{n+\frac{9}{8}}-\frac{1440464}{42525}h^2f_{n+2}+\frac{10883072}{297675}h^2f_{n+\frac{17}{8}}\\
&-\frac{5064128}{253575}h^2f_{n+3}+\frac{501911552}{25302375}h^2f_{n+\frac{25}{8}}+\frac{52804888}{70747425}h^2f_{n+4}
\end{aligned}\right\}\quad(43)$$

$$\left.\begin{aligned}
&p_{n+\frac{1}{8}}\\
&=p_{n+4}-\frac{31}{8}hq_{n+4}-\frac{21252721875431}{30321672192000}h^2f_n+\frac{243267367981}{199419494400}h^2f_{n+\frac{1}{8}}\\
&-\frac{25531487020531}{2122517053440}h^2f_{n+1}+\frac{28586806638629}{1794775449600}h^2f_{n+\frac{9}{8}}-\frac{4562347968007}{138726604800}h^2f_{n+2}\\
&+\frac{1879041079129}{53062926336}h^2f_{n+\frac{17}{8}}-\frac{92541069943499}{4786067865600}h^2f_{n+3}+\frac{14134885095509}{736985088000}h^2f_{n+\frac{25}{8}}\\
&+\frac{552788310287}{765770858496}h^2f_{n+4}
\end{aligned}\right\} \tag{44}$$

$$\left.\begin{aligned}
&p_{n+1}\\
&=p_{n+4}-3hq_{n+4}-\frac{45963}{95200}h^2f_n+\frac{714312}{873425}h^2f_{n+\frac{1}{8}}-\frac{18881}{1960}h^2f_{n+1}+\frac{340056}{28175}h^2f_{n+\frac{9}{8}}\\
&-\frac{498249}{19600}h^2f_{n+2}+\frac{559768}{20825}h^2f_{n+\frac{17}{8}}-\frac{3335553}{225400}h^2f_{n+3}+\frac{17784}{1225}h^2f_{n+\frac{25}{8}}\\
&+\frac{15394311}{27949600}h^2f_{n+4}
\end{aligned}\right\} \tag{45}$$

$$\left.\begin{aligned}
&p_{n+\frac{9}{8}}\\
&=p_{n+4}-\frac{23}{8}hq_{n+4}-\frac{4637748085277}{10107224064000}h^2f_n+\frac{626650104859}{806348390400}h^2f_{n+\frac{1}{8}}\\
&-\frac{19371400801603}{2122517053440}h^2f_{n+1}+\frac{59280222977}{5202247680}h^2f_{n+\frac{9}{8}}-\frac{30327747946607}{1248539443200}h^2f_{n+2}\\
&+\frac{33967943740933}{1326573158400}h^2f_{n+\frac{17}{8}}-\frac{65388852593}{4624220160}h^2f_{n+3}+\frac{13118795805383}{947552256000}h^2f_{n+\frac{25}{8}}\\
&+\frac{1940407925977}{3686164070400}h^2f_{n+4}
\end{aligned}\right\} \tag{46}$$

$$\left.\begin{aligned}
&p_{n+2}\\
&=p_{n+4}-2hq_{n+4}-\frac{2141519}{7229250}h^2f_n+\frac{514048}{1025325}h^2f_{n+\frac{1}{8}}-\frac{5886248}{1012095}h^2f_{n+1}+\frac{2148352}{297675}h^2f_{n+\frac{9}{8}}\\
&-\frac{1585459}{99225}h^2f_{n+2}+\frac{16603136}{1012095}h^2f_{n+\frac{17}{8}}-\frac{943064}{99225}h^2f_{n+3}+\frac{10983424}{1204875}h^2f_{n+\frac{25}{8}}+\frac{62549}{175770}h^2f_{n+4}
\end{aligned}\right\} \tag{47}$$

$$\left.\begin{aligned}
&p_{n+\frac{17}{8}}\\
&=p_{n+4}-\frac{15}{8}hq_{n+4}-\frac{16067025}{58720256}h^2f_n+\frac{4240913625}{9158524928}h^2f_{n+\frac{1}{8}}-\frac{9377358125}{1746927616}h^2f_{n+1}\\
&+\frac{1967628375}{295436288}h^2f_{n+\frac{9}{8}}-\frac{3020374125}{205520896}h^2f_{n+2}+\frac{193203275}{12845056}h^2f_{n+\frac{17}{8}}-\frac{20886973875}{2363490304}h^2f_{n+3}\\
&+\frac{1842390675}{218365952}h^2f_{n+\frac{25}{8}}+\frac{97157995425}{293072797696}h^2f_{n+4}
\end{aligned}\right\} \tag{48}$$

$$\left.\begin{array}{l} p_{n+3} \\ = p_{n+4} - hq_{n+4} - \frac{4615363}{38556000}h^2 f_n + \frac{14325368}{70747425}h^2 f_{n+\frac{1}{8}} - \frac{18910417}{8096760}h^2 f_{n+1} + \frac{264232}{91287}h^2 f_{n+\frac{9}{8}} \\ - \frac{29864843}{4762800}h^2 f_{n+2} + \frac{32206072}{5060475}h^2 f_{n+\frac{17}{8}} - \frac{4857709}{1217160}h^2 f_{n+3} + \frac{90935864}{25302375}h^2 f_{n+\frac{25}{8}} \\ + \frac{365637323}{2263917600}h^2 f_{n+4} \end{array}\right\} \quad (49)$$

$$\left.\begin{array}{l} p_{n+\frac{25}{8}} \\ = p_{n+4} - \frac{7}{8}hq_{n+4} - \frac{426389289809}{4331667456000}h^2 f_n + \frac{677592041}{4069785600}h^2 f_{n+\frac{1}{8}} - \frac{4889332259}{2548039680}h^2 f_{n+1} \\ + \frac{87091077521}{36628070400}h^2 f_{n+\frac{9}{8}} - \frac{43666584661}{8493465600}h^2 f_{n+2} + \frac{141204004669}{27072921600}h^2 f_{n+\frac{17}{8}} - \frac{316262600171}{97674854400}h^2 f_{n+3} \\ + \frac{7649495987}{2654208000}h^2 f_{n+\frac{25}{8}} + \frac{53640435191}{390699417600}h^2 f_{n+4} \end{array}\right\} \quad (50)$$

Obtaining the fourth derivative of (9) and evaluating at $x = x_n, x_{n+\frac{1}{8}}, x_{n+1}, x_{n+\frac{9}{8}}, x_{n+2}, x_{n+\frac{17}{8}}, x_{n+3}, x_{n+\frac{25}{8}}$ gives the following discrete schemes:

$$\left.\begin{array}{l} q_n \\ = q_{n+4} + \frac{1113914}{3614625}hf_n - \frac{62291968}{70747425}hf_{n+\frac{1}{8}} + \frac{1616192}{1012095}hf_{n+1} - \frac{22970368}{6846525}hf_{n+\frac{9}{8}} \\ + \frac{2352088}{297675}hf_{n+2} - \frac{6846525}{103275}hf_{n+\frac{17}{8}} + \frac{236864}{46575}hf_{n+3} - \frac{133070848}{25302375}hf_{n+\frac{25}{8}} \\ - \frac{13873606}{70747425}hf_{n+4} \end{array}\right\} \quad (51)$$

$$\left.\begin{array}{l} q_{n+\frac{1}{8}} \\ = q_{n+4} + \frac{685571673401}{1895104512000}hf_n - \frac{120558952229}{149564620800}hf_{n+\frac{1}{8}} + \frac{207687271307}{132657315840}hf_{n+1} \\ - \frac{64814580613}{19508428800}hf_{n+\frac{9}{8}} + \frac{614575549661}{78033715200}hf_{n+2} - \frac{434860031357}{47377612800}hf_{n+\frac{17}{8}} \\ + \frac{217134176153}{217134176153}hf_{n+3} - \frac{8715841719779}{1658216448000}hf_{n+\frac{25}{8}} - \frac{10203022343}{52022476800}hf_{n+4} \end{array}\right\} \quad (52)$$

$$\left.\begin{array}{l} q_{n+1} \\ = q_{n+4} + \frac{45581}{238000}hf_n - \frac{284136}{873425}hf_{n+\frac{1}{8}} + \frac{401057}{99960}hf_{n+1} - \frac{152632}{28175}hf_{n+\frac{9}{8}} \\ + \frac{22159}{2450}hf_{n+2} - \frac{90968}{8925}hf_{n+\frac{17}{8}} + \frac{169473}{32200}hf_{n+3} - \frac{560312}{104125}hf_{n+\frac{25}{8}} - \frac{2726207}{13974800}hf_{n+4} \end{array}\right\} \quad (53)$$

$$\left.\begin{aligned}&q_{n+\frac{9}{8}}\\&=q_{n+4}+\frac{363246259513}{1895104512000}hf_{n}-\frac{65638309949}{201587097600}hf_{n+\frac{1}{8}}+\frac{401540138062731057}{132657315840}hf_{n+1}\\&-\frac{104384340869}{19508428800}hf_{n+\frac{9}{8}}+\frac{705433644253}{78033715200}hf_{n+2}-\frac{482723814781}{47377612800}hf_{n+\frac{17}{8}}+\frac{9777595343}{1857945600}hf_{n+3}\\&-\frac{8922486713827}{1658216448000}hf_{n+\frac{25}{8}}-\frac{314609974489}{1612696780800}hf_{n+4}\end{aligned}\right\} \quad (54)$$

$$\left.\begin{aligned}&q_{n+2}\\&=q_{n+4}+\frac{1307429}{7229250}hf_{n}-\frac{942208}{3075975}hf_{n+\frac{1}{8}}+\frac{3626506}{1012095}hf_{n+1}-\frac{1326208}{297675}hf_{n+\frac{9}{8}}+\frac{3040844}{297675}hf_{n+2}\\&-\frac{7901312}{722925}hf_{n+\frac{17}{8}}+\frac{75638}{14175}hf_{n+3}-\frac{137377664}{25302375}hf_{n+\frac{25}{8}}-\frac{1198007}{6151950}hf_{n+4}\end{aligned}\right\} \quad (55)$$

$$\left.\begin{aligned}&q_{n+\frac{17}{8}}\\&=q_{n+4}+\frac{11286815}{62390272}hf_{n}-\frac{701556675}{2289631232}hf_{n+\frac{1}{8}}+\frac{1174153375}{327548928}hf_{n+1}-\frac{329206825}{73859072}hf_{n+\frac{9}{8}}\\&+\frac{132040775}{12845056}hf_{n+2}-\frac{36324515}{3342336}hf_{n+\frac{17}{8}}+\frac{16083975}{3014656}hf_{n+3}-\frac{296374885}{54591488}hf_{n+\frac{25}{8}}\\&-\frac{3567045415}{18317049856}hf_{n+4}\end{aligned}\right\} \quad (56)$$

$$\left.\begin{aligned}&q_{n+3}\\&=q_{n+4}+\frac{578623}{3402000}hf_{n}-\frac{20361832}{70747425}hf_{n+\frac{1}{8}}+\frac{26977609}{8096760}hf_{n+1}-\frac{1230104}{297675}hf_{n+\frac{9}{8}}\\&+\frac{5375099}{595350}hf_{n+2}-\frac{55784}{6075}hf_{n+\frac{17}{8}}+\frac{2227903}{372600}hf_{n+3}-\frac{144561832}{25302375}hf_{n+\frac{25}{8}}\\&-\frac{9532673}{49215600}hf_{n+4}\end{aligned}\right\} \quad (57)$$

$$\left.\begin{aligned}&q_{n+\frac{25}{8}}\\&=q_{n+4}+\frac{46089615551}{270729216000}hf_{n}-\frac{27259019627}{94622515200}hf_{n+\frac{1}{8}}+\frac{9029577851}{2707292160}hf_{n+1}-\frac{37879152227}{9157017600}hf_{n+\frac{9}{8}}\\&+\frac{14395807853}{1592524800}hf_{n+2}-\frac{62230072667}{6768230400}hf_{n+\frac{17}{8}}+\frac{36930847583}{6104678400}hf_{n+3}-\frac{191434980947}{33841152000}hf_{n+\frac{25}{8}}\\&-\frac{146641270447}{756980121600}hf_{n+4}\end{aligned}\right\} \quad (58)$$

where $z_{n+j}=y'_{n+j}, g_{n+j}=y''_{n+j}, p_{n+j}=y'''_{n+j}, q_{n+j}=y^{iv}_{n+j}, j=0,\frac{1}{8},1,\frac{9}{8},2,\frac{17}{8},3,\frac{25}{8}.$

2.1. Analysis of the Fundamental Characteristics of the block Method

The attributes of the block method are scrutinized as follows:

2.2 *Order and Error Constants of the block method*

The technique of Lambert (1973) and Fatunla (1991) for determining the order of a numerical discrete schemes is used to find the order of the new block method. Hence, the new block method is of equal order $p = 9$, that is $p=[9,9,9,9,9,9,9,9]^T$ with error constants

$$\left[,\frac{24233079470933902847}{123695058124800884801952591158181888OO},\frac{4620211}{51661209600},\frac{5678507831963763253}{7668283589123370909696O},\frac{60458487671}{4284987369062400},\frac{21015905896442389}{202864645214903992320O},-\frac{89752236313}{285665824604160},-\frac{1471450636315149108139 9}{5656728483336866478489600O0},-\frac{28415417}{52901078630400}\right]^T$$

2.3 *Zero stability of the block method*

Definition 1: (c.f. Fatunla 1988): A block method is said to be zero-stable if the roots r_j of the first characteristic polynomial $\rho(r) = |rA^1 - A^0| = 0$ and $\rho(r) = 0$ satisfies $|r_j| \leq 1, j = 1,2,3 \ldots$ where

$$A^{(0)} = \begin{bmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&-1 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{bmatrix}, A^{(1)} = \begin{bmatrix} 1&0&0&0&0&-1&0&0 \\ 0&1&0&0&0&-1&0&0 \\ 0&0&1&0&0&-1&0&0 \\ 0&0&0&1&0&-1&0&0 \\ 0&0&0&0&1&-1&0&0 \\ 0&0&0&0&0&1&0&-1 \\ 0&0&0&0&0&-1&1&0 \\ 0&0&0&0&0&-1&0&1 \end{bmatrix}$$ are the coefficients of

$\left[y_{n-4}, y_{n-\frac{25}{8}}, y_{n-3}, y_{n-\frac{17}{8}}, y_{n-\frac{9}{8}}, y_{n-1}, y_{n-\frac{1}{8}}, y_{n}\right]^T$ and $\left[y_{n+4}, y_{n+\frac{25}{8}}, y_{n+3}, y_{n+\frac{17}{8}}, y_{n+\frac{9}{8}}, y_{n+\frac{1}{8}}, y_{n+2}, y_{n+1}\right]^T$ respectively

Normalizing $A^{(1)}$ and $A^{(0)}$ we obtained

$$A^0 = \begin{bmatrix} 0&0&0&0&0&0&0&-1 \\ 0&0&0&0&0&0&0&-1 \\ 0&0&0&0&0&0&0&-1 \\ 0&0&0&0&0&0&0&-1 \\ 0&0&0&0&0&0&0&-1 \\ 0&0&0&0&0&0&0&-1 \\ 0&0&0&0&0&0&0&-1 \\ 0&0&0&0&0&0&0&-1 \end{bmatrix} \text{ and } A^1 = \begin{bmatrix} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \end{bmatrix}$$

$\Rightarrow \rho(r) = |rA^1 - A^0| = 0$

$\Rightarrow \rho(r) = r^7(r+1) = 0$

$\Rightarrow |r| = (0, 0, 0, 0, 0,0,0, 1)$.

Therefore, the block method exhibits zero stability as it meets $|r_j| \leq 1, j = 1,2,3,4,5,6,7,8$.

2.4 *Consistency*

Definition 2: (c.f. Lambert 1973): A linear multistep method is said to be consistent if the following conditions are satisfied:

i) The order p of accuracy is greater than one ($p > 1$)

ii) $\sum_{j=0}^{k} \alpha_j = 0$.

It is evident that the block method maintains consistency since $p = 9$, $\Rightarrow$ p $>$ 1and
$\alpha_0 + \alpha_{\frac{1}{8}} + \alpha_1 + \alpha_{\frac{9}{8}} + \alpha_2 + \alpha_{\frac{17}{8}} + \alpha_3 + \alpha_{\frac{25}{8}} + \alpha_4 = 0$

2.5 *Convergence*

Definition 3: (c.f. Henrici 1962, Lambert 1973, Kuboye and Adeyefa 2021): The necessary and sufficient conditions for a linear multistep method to be convergent is to be consistent and zero stable. Following this definition or theorem, the block method developed is convergent.

2.6 *Region of Absolute Stability*

Following Akinfenwa *et al*, (2014), the region of absolute stability is defined by obtaining the stability polynomial of the form:

$$\sigma(z) = -\left(A(1) - zB(1) - z^2C(1) - z^3E(1) - z^4F(1) - z^5G(1)\right)^{-1}\left(A(0) - z^5G(0)\right) \quad (59)$$

Where $z = \lambda$h and A(0), A(1), B(1), C(1), E(1), F(1), G(0), G(1) are the coefficients of the main discrete schemes of the proposed method (20-27).

The matrix $\sigma(z)$ has eigenvalues $\{0,0,0, \ldots \lambda_k\}$ and the dominant eigenvalue $\lambda_k: \mathbb{C} \rightarrow \mathbb{C}$ is a rational function (called the stability function) with real coefficients given by

$$\lambda_k = \frac{P(z)}{P(-z)}$$

The stability polynomial (59) was generated and plotted using Maple software which produced the required region of absolute stability of the method below.

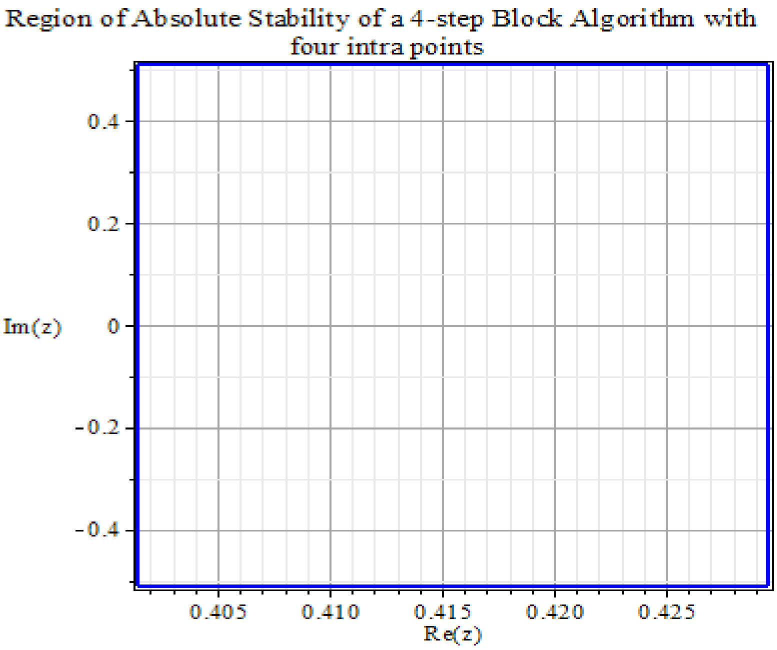

Figure 1: *Stability region for a 4-Step Block Algorithm with Four intra points*

3. RESULT AND DISCUSSION

This section presents numerical examples to demonstrate the effectiveness of the proposed method. We calculate the absolute errors of the approximate solution and also compare it with some existing methods in the literature.

Problem 1 (SIR model): the SIR model is an epidemiological model that computes the theoretical number of people infected with a contagious illness in a closed population over time. The name of this class of models derives from the fact that they involve coupled equations relating the number of susceptible people S(t), no of people infected I (t) and the number of people who have recovered R (t). This is a good and simple model for many infectious diseases including measles, mumps and rubella. It is given by the following three coupled equations:

$$\frac{dS}{dt} = \mu(I - S) - \beta IS \tag{60}$$

$$\frac{dS}{dt} = \mu I - \gamma I + \beta IS \tag{61}$$

$$\frac{dR}{dt} = -\mu R + \gamma I \tag{62}$$

Where μ, β *and* γ are parameters that are positive. Let y be defined as:
$y = S + I + R$. Adding (60)-(62), the following evolution equation for y is obtained.

$y' = \mu(1 - y)$. Taking $\mu = 0.5$ and attaching an initial condition $y(0) = 0.5$ (for a particular closed population), the equation below is achieved.
$y'(t) = 0.5(1 - y), y(0) = 0.5, h = 0.1.$
Exact Solution $y(t) = 1 - 0.5e^{-0.5t}$
Source: John *et al.* (2022).

Problem 2: Real-life problem (cooling of body).
The temperature y degrees of a body, t minutes after being placed in a certain room, satisfies the differential equation $3\frac{d^2y}{dt^2} + \frac{dy}{dt} = 0$. By using the substitution $z = \frac{dy}{dt}$ or employing another method, determine y in relation to t, considering that $y\ equals$ 60 when t is 0 and $y\ is$ 35 when $t\ is\ 6ln4$. Calculate the duration, rounded to the nearest minute, when the rate of cooling of the body drops below one degree per minute. The problem's formulation.
$y'' = \frac{-y'}{3}, y(0) = 60, y'(0) = -\frac{80}{9}, h = 0.1$ precise solution: $y(t) = \frac{80}{3}e^{\left(\frac{1}{3}\right)t} + \frac{100}{3}$
Source: Ezekiel *et al*. (2018).

Problem 3: Non-linear third order ODE problem:
$y''' - xy'' + x^2y^2 = xsinx - cosx + x^2sin^2x, y(0) = 0, y'(0) = 1, y''(0) = 0, x \in [0,1]$.
precise Solution $\boldsymbol{y(x) = sinx, h = 0.1}$
Source: Oluwaseun and Zurni (2019)

Problem 4: Non-linear 4th order ODE problem:
$y^{iv} = (y')^2 - yy''' - 4x^2 + e^x(1 - 4x) + x^2, y(0) = 1, y'(0) = 1, y''(0) = 3, y'''(0) = 1$
precise solution: $\boldsymbol{y(x) = x^2 + e^x}$
Source: Victor (2021) and Olusola (2022).

Problem 5: Consider linear fifth order problem
$y^v = 32y + cosx - 32sinx, y(0) = 1, y'(0) = 3, y''(0) = 4, y'''(0) = 7, y^{iv}(0) = 16, h = 0.1$
precise solution: $y(x) = sinx + e^{2x}$
Source: Roseline *et al*. (2021).

Table 1: *Comparison of the precise solution with the suggested approach for Problem 1*

x	Precise Solution	Estimated Solution	Absolute Error
0.1	0.524385287749642995454287340110	0.524385287749642995454229567099	5.78×10^{-23}
0.2	0.547581290982020213417875470277	0.547581290982020213417800946144	7.45×10^{-23}
0.3	0.569646011787471096385483117728	0.569646011787471096385408533434	7.46×10^{-23}
0.4	0.590634623461009070665032245690	0.590634623461009070664961158019	7.11×10^{-23}
0.5	0.610599608464297565877414866511	0.610599608464297565877299945286	1.15×10^{-22}
0.6	0.629590889659141066966563110341	0.629590889659141066966437772357	1.25×10^{-22}

0.7	**0.647655955140643282822589650484**	**0.647655955140643282822467400305**	**1.22**$\times 10^{-22}$
0.8	**0.664839976982180349627783537426**	**0.664839976982180349627667134099**	**1.22**$\times 10^{-22}$
0.9	**0.681185924189113353428128280844**	**0.681185924189113353427978828168**	**1.49**$\times 10^{-22}$
1.0	**0.696734670143683288198100232504**	**0.696734670143683288197944951398**	**1.55**$\times 10^{-22}$

Table 2: *Comparison of the precise solution with the suggested approach for Problem 2*

x	Precise Solution	Estimated Solution	Absolute Error
0.1	**59.1257626795201573877592829158**	**59.1257626795201573877592772358**	**5.6800**$\times 10^{-24}$
0.2	**58.2801862675098063394782344439**	**58.2801862675098063394782158650**	**1.8579**$\times 10^{-23}$
0.3	**57.4623311476255886177133082519**	**57.4623311476255886177132741351**	**3.4116**$\times 10^{-23}$
0.4	**56.6712885078119321065183306579**	**56.6712885078119321065182808293**	**4.9828**$\times 10^{-23}$
0.5	**55.9061793304163753078644639946**	**55.9061793304163753078643939714**	**7.0023**$\times 10^{-23}$
0.6	**55.1661534154128495645316135631**	**55.1661534154128495645315175265**	**9.6037**$\times 10^{-23}$
0.7	**54.4503884356475110501206420451**	**54.4503884356475110501205181682**	**1.2388**$\times 10^{-22}$
0.8	**53.7580890230572984723111375205**	**53.7580890230572984723109861180**	**1.5140**$\times 10^{-22}$
0.9	**53.0884858848458097617833007818**	**53.0884858848458097617831183821**	**1.8239**$\times 10^{-22}$
1.0	**52.4408349486343800113494425847**	**52.4408349486343800113492245321**	**2.1805**$\times 10^{-22}$

Table 3: *Comparison of the precise solution with the suggested approach for Problem 3*

x	Precise Solution	Estimated Solution	Absolute Error
0.1	**0.099833416646828152306814198410**	**0.099833416646828152306814198410**	**0.0**$\times 10^{00}$
0.2	**0.198669330795061215459412627118**	**0.198669330795061215459412627118**	**0.0**$\times 10^{00}$
0.3	**0.295520206661339575105320745685**	**0.295520206661339575105320745682**	**2.6**$\times 10^{-30}$
0.4	**0.389418342308650491666311756796**	**0.389418342308650491666311756788**	**8.1**$\times 10^{-30}$
0.5	**0.479425538604203000273287935216**	**0.479425538604203000273287935196**	**1.9**$\times 10^{-29}$
0.6	**0.564642473395035357200945445659**	**0.564642473395035357200945445618**	**4.1**$\times 10^{-29}$
0.7	**0.644217687237691053672614351399**	**0.644217687237691053672614351324**	**7.5**$\times 10^{-29}$
0.8	**0.717356090899522761627174610581**	**0.717356090899522761627174610452**	**1.9**$\times 10^{-28}$
0.9	**0.783326909627483388461382315713**	**0.783326909627483388461382315506**	**2.1**$\times 10^{-28}$
1.0	**0.841470984807896506652502321630**	**0.841470984807896506652502321310**	**3.2**$\times 10^{-28}$

Table 4: *Comparison of the precise solution with the suggested approach for Problem 4*

x	Precise Solution	Estimated Solution	Absolute Error
0.1	**1.11517091807564762481170782649**	**1.11517091807564762481170782648857**	**1.4**$\times 10^{-30}$
0.2	**1.26140275816016983392107199464**	**1.26140275816016983392107199461383**	**2.6**$\times 10^{-29}$
0.3	**1.43985880757600310398374431333**	**1.43985880757600310398374431319901**	**1.3**$\times 10^{-28}$
0.4	**1.65182469764127031782485295284**	**1.65182469764127031782485295243335**	**4.1**$\times 10^{-28}$
0.5	**1.89872127070012814684865078781**	**1.89872127070012814684865078683603**	**9.7**$\times 10^{-28}$
0.6	**2.18211880039050897487536766816**	**2.18211880039050897487536766615026**	**2.0**$\times 10^{-27}$
0.7	**2.50375270747047652162454938858**	**2.50375270747047652162454938488370**	**3.7**$\times 10^{-27}$
0.8	**2.86554092849246760457953753140**	**2.86554092849246760457953752513515**	**6.3**$\times 10^{-27}$
0.9	**3.26960311115694966380012656360**	**3.26960311115694966380012655365854**	**9.9**$\times 10^{-27}$
1.0	**3.71828182845904523536028747135**	**3.71828182845904523536028745632406**	**1.5**$\times 10^{-26}$

Table 5: *Comparison of the precise solution with the suggested approach for Problem 5*

x	Precise Solution	Estimated Solution	Absolute Error
0.1	1.3212361748069979862	1.3212361748069979905	4.3000×10^{-18}
0.2	1.6904940284363315333	1.6904940284363317557	2.2240×10^{-16}
0.3	2.1176390070518485500	2.1176390070518502529	1.7029×10^{-15}
0.4	2.6149592708011180963	2.6149592708011248324	6.7361×10^{-15}
0.5	3.1977073670632482357	3.1977073670632685724	2.0337×10^{-14}
0.6	3.8847593961315828467	3.8847593961316389714	5.6125×10^{-14}
0.7	4.6994176540823656409	4.6994176540825063368	1.4069×10^{-13}
0.8	5.6703885152946375653	5.6703885152949522257	3.1466×10^{-13}
0.9	6.8329743740404294722	6.8329743740410654484	6.3597×10^{-13}
1.0	8.2305270837385467338	8.2305270837397398393	1.1931×10^{-12}

Table 6: *Comparison of the error of the suggested approach for Problem 1 with John et al (2022)*

x	Precise Solution	Error in suggested approach k = 4, h = 0.1 , p = 9	Error in John *et al.* (2022) k = ½ , h = 0.1, p = 6
0.1	0.524385287749642995454287340110	5.78×10^{-23}	4.939×10^{-13}
0.2	0.547581290982020213417875470277	7.45×10^{-23}	9.399×10^{-13}
0.3	0.569646011787471096385483117728	7.46×10^{-23}	1.341×10^{-12}
0.4	0.590634623461009070665032245690	7.11×10^{-23}	1.701×10^{-12}
0.5	0.610599608464297565877414866511	1.15×10^{-22}	2.022×10^{-12}
0.6	0.629590889659141066966563110341	1.25×10^{-22}	2.308×10^{-12}
0.7	0.647655955140643282822589650484	1.22×10^{-22}	2.592×10^{-12}
0.8	0.664839976982180349627783537426	1.22×10^{-22}	………………………
0.9	0.681185924189113353428128280844	1.49×10^{-22}	………………………
1.0	0.696734670143683288198100232504	1.55×10^{-22}	………………………

Table 7: *Comparison of the error of the suggested approach for Problem 2 with Ezekiel et al. (2018)*

x	Precise Solution	Error in suggested approach $k = 4, h = 0.1, p = 9$	Error in Ezekiel et al (2018) $k = 3, h = 0.1, p = 5$
0.1	59.1257626795201573877592829158	5.6800×10^{-24}	3.55×10^{-11}
0.2	58.2801862675098063394782344439	1.8579×10^{-23}	4.58×10^{-11}
0.3	57.4623311476255886177133082519	3.4116×10^{-23}	7.00×10^{-11}
0.4	56.6712885078119321065183306579	4.9828×10^{-23}	6.50×10^{-12}
0.5	55.9061793304163753078644639946	7.0023×10^{-23}	3.33×10^{-11}
0.6	55.1661534154128495645316135631	9.6037×10^{-23}	4.20×10^{-11}
0.7	54.4503884356475110501206420451	1.2388×10^{-22}	4.38×10^{-11}
0.8	53.7580890230572984723111375205	1.5140×10^{-22}	1.07×10^{-10}
0.9	53.0884858848458097617833007818	1.8239×10^{-22}	6.58×10^{-11}
1.0	52.4408349486343800113494425847	2.1805×10^{-22}	1.69×10^{-10}

Table 8: *Comparison of the error of the suggested approach for Problem 3 with Oluwaseun (2019)*

x	Precise Solution	Error in suggested approach $k = 4, h = 0.1, p = 9$	Error in Oluwaseun (2019) $k = 1, h = 0.1, p = 5$
0.1	0.0998334166468281523068141984106	4.3×10^{-32}	6.7×10^{-16}
0.2	0.198669330795061215459412627118	5.6×10^{-31}	3.9×10^{-15}
0.3	0.295520206661339575105320745685	2.6×10^{-30}	1.2×10^{-14}
0.4	0.389418342308650491666311756796	8.1×10^{-30}	2.9×10^{-14}
0.5	0.479425538604203000273287935216	1.9×10^{-29}	5.6×10^{-14}
0.6	0.564642473395035357200945445659	4.1×10^{-29}	9.7×10^{-14}
0.7	0.644217687237691053672614351399	7.5×10^{-29}	1.5×10^{-13}
0.8	0.717356090899522761627174610581	1.9×10^{-28}	2.3×10^{-13}
0.9	0.783326909627483388461382315713	2.1×10^{-28}	3.3×10^{-13}
1.0	0.841470984807896506652502321630	3.2×10^{-28}	4.6×10^{-13}

Table 9: *Comparison of the error of the suggested approach for Problem 4 with Victor (2021)*

x	Precise Solution	Error in suggested approach k = 4, h = 0.1, p = 9	Error in Victor (2021) k = 15, h = 0.003125, p = 16
0.1	1.11517091807564762481170782649	1.4×10^{-30}	2.220×10^{-16}
0.2	1.26140275816016983392107199464	2.6×10^{-29}	$0.000 \times 10^{+00}$
0.3	1.43985880757600310398374431333	1.3×10^{-28}	2.220×10^{-16}
0.4	1.65182469764127031782485295284	4.1×10^{-28}	$0.000 \times 10^{+00}$
0.5	1.89872127070012814684865078781	9.7×10^{-28}	$0.000 \times 10^{+00}$
0.6	2.18211880039050897487536766816	2.0×10^{-27}	2.220×10^{-16}
0.7	2.50375270747047652162454938858	3.7×10^{-27}	6.661×10^{-16}
0.8	2.86554092849246760457953753140	6.3×10^{-27}	1.332×10^{-15}
0.9	3.26960311115694966380012656360	9.9×10^{-27}	2.887×10^{-15}
1.0	3.71828182845904523536028747135	1.5×10^{-26}	……………………..

The results of the four-step approach, developed to solve linear and nonlinear ODEs from first to fifth orders, are presented in Tables 1 to 5. The errors were calculated as the absolute difference between the exact and estimated solutions. The minimal errors observed in the new approach demonstrate its effectiveness and efficiency.

Tables 6 and 7 provide a comparison of the outcomes of the new hybrid block method with those obtained by John (2022) and Ezekiel (2018) for problems 1 and 2, respectively. The results show that the new method outperforms both existing methods in terms of error. Furthermore, tables 8 and 9 reveal that the new method significantly surpasses the outcomes achieved by Oluseun (2019) and Victor (2021) for problems 3 and 4, respectively. This comparison highlights the superior performance of the new hybrid block method in solving ODEs of different orders.

4. CONCLUSION

A novel hybrid block technique has been designed for solving ODEs of order up to five, with a specific focus on step number $k = 4$. This approach combines both continuous and discrete schemes, derived using interpolation and collocation methods, resulting in high accuracy of 9th order for $k = 4$. The method's fundamental properties were examined, and it was found to meet the requirements for consistency, zero stability, and convergence. To evaluate the effectiveness of the new method, test examples were run on various linear and non-linear IVPs of ODEs up to fifth order. The results, presented in Tables 6 to 9, demonstrate superior accuracy compared to existing methods. One of the key benefits of this approach is its ability to directly tackle IVPs of ODEs ranging from first to fifth order, eliminating the need to develop separate methods for each order. This new method can significantly reduce the time required to develop numerical methods for ODEs of different orders, as it can directly handle first through fifth-order IVPs without the need for additional formulation. As a result, this method is highly recommended as an efficient and effective numerical method for solving IVPs of ODEs.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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