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Thickening annulus in Cauchy's integral Theorem for matrix function with Mobius Transformation via Trapezoidal rule and Runge-Kutta fourth –order method

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Abstract

This study discusses thickening annulus in Cauchy's integral theorem for matrix function via Mobius transformation. We introduce a Jacobi type elliptic integral for arc length using Mobius transformation leading to optimization in the contour integral wherein Trapezoidal rule and Runge-Kutta fourth order method are used yielding approximate solutions to the integral problem. We filtered out noise from solution space using Tikhonov regularization method. This leads to computing for the path integral, path length and size of the path for the Cauchy integral. We have established that integral operators used have analytic continuation in their paths. As a further insight into what was studied is the computation of action matrix on Cauchy integral for each of Trapezoidal rule and Runge Kutta method.

Keywords: Cauchy's integral theorem, Jacobi elliptic function, Tikhonov regularization, action matrix
AMS subject classification:primary 28A20, 28A25, 30E20, Secondary: 65F15, 65F60.

1 Introduction

We often use numerical methods in computing Cauchy's integral theorem by Trapezoid rule Futamura and Sakurai (2021), Hale *et al* (2008), Higham (2008), Suli (2013), Uwamusi (2015). However, the process which produces these results may be affected by distortion in rotation as the closed contour lying in the region of analyticity of the measurable f swings around once around the spectrum $\sigma(A)$ in the counterclockwise direction. In this paper, we have made a successful effort in introducing a new process of computation leading to thickening annulus in Cauchy's integral theorem for matrix function. If the function $f(z)$ is holomorphic in a region of analyticity, we then use the largest and smallest eigenvalues in place of these radii in Cauchy integral theorem for the matrix function since the radii R and r are not available apriori. In the complex realm, Cauchy's integral theorem for the matrix function Futamura and Sukurai (2021), Hale *et al* (2008) and Higham (2008) is expressed in the form:

$$f(A) = \frac{1}{2\pi i}\int_{\eta} \frac{f(z)}{(zI - A)} dz \tag{1}$$

In equation (1), the integral ranges over contour η in a counterclockwise direction where $z = re^{i\theta}, r = |z|, A \in \Re^{m\times n}$ or $A \in \mathbb{C}^{m\times n}$ and I is an identity matrix of same dimension with matrix A. That $A \in R^{n\times n}$ is a common place in everyday occurrence. It is supposed that eigenvalues of A are located in R^{+} . Following Uwamusi (2022) the term $(zI - A)^{-1}$ is expanded in the form

$$(zI - A)^{-1} = \frac{1}{z}\left(I + \frac{A}{z} + \frac{A^2}{z^2} + ... + \frac{A^n}{z^n} + ... \right) \tag{2}$$

For convergence, it is necessary that the term $\left\| \frac{A}{z} \right\| < 1$ and this yields

$$\frac{f(z)}{(zI - A)} = \frac{f(z)}{z} I + \frac{Af(z)}{z^2} + \frac{A^2 f(z)}{z^3} + ... + \frac{A^n F(z)}{z^{n+1}} + ... \tag{3}$$

Because of equation (3), we rewrite equation (1) in terms of Taylor series expansion Taylor (2018) :

$$\frac{1}{2\pi i}\oint_{\eta} \frac{f(z)}{zI - A} dz = \frac{1}{2\pi i}\oint_{\eta} \frac{f(z)}{z} I dz + \frac{A}{2\pi i}\oint_{\eta} \frac{f(z)}{z^2} dz + \frac{A^2}{2\pi i}\oint_{\eta} \frac{f(z)}{z^3} + ... + \frac{A^n}{2\pi i}\oint_{\eta} \frac{f(z)}{z^{n+1}} dz + ...$$

It is usual to modify equation (1). Such a modification was obtained in equation Futamura and Sukurai (2021), Hale *et al* (2008) in the equivalent form

$$f(A) = \frac{A}{2\pi i}\oint_{\eta} z^{-1} f(z)(zI - A)^{-1} dz\,. \tag{4}$$

The introduced term z^{-1} was to stabilize the level of distortion dz when Jacobi elliptic integral function is applied in the calculation.

One important thing to note is that newer methods and innovations are emerging in the calculation of Cauchy's integral theorem, see e.g., Futamura and Sukurai (2021), Uwamusi (2015). We define the quantity $4\left(\frac{|f'(A)|}{I + |f(A)|^2} \right)^2 I$ as representing the spherical derivative in the complex holomorphic function $f(A)$ as applicable in Riemann surface problems Lawden (1989), Taylor (2018).

We compute the path integral (contour integral) in the complex measureable function $f(A)$ as follows: Given that there is a map $\eta : [a,b] \to \mathbb{C}$ an oriented differentiable C'-curve in which $f : \eta^{*} \to \mathbb{C}$ is continuous in the matrix function $f(A)$, the path integral in Cauchy's integral theorem is then defined in the form

$$\oint_{\eta} f = \int_{\eta} f(A)dz = \int_{a}^{b} f(\eta(A))\eta^{/}(A)dz, \; z \in [a,b]. \quad (5)$$

The concept of parameterization of a path integral is an important aspect in the complex analysis. Thus, by a parameterization of a path, we mean a continuous partition over a measurable space $\eta:[a,b] \to \mathbb{C}$ which is piecewise $C^{/}$ in which exists , a N-slices such that

$a = z_0 < z_1 < ... < z_{n-1} < z_n = b$ and that, $\eta_k = \eta|z_{k-1}, z_k|, k = 1,2,3,...,n$ are $C^{/}$ - parameterization.

Definition 1, Taylor (2018). *By the term integral of a complex measurable function* f *along a path* η*, we mean a complex number ranging over N- slices such that:*

$$\int_{\eta} f = \int_{\eta_1 \cup \eta_2 \cup .. \eta_n} f = \sum_{k=1}^{n} \int_{\eta_k} f = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} f(\eta(A))\eta^{/}(A)dz.$$

We define the length of a path $\eta:[a,b] \to \mathbb{C}$ *in the equation (1.1) when* $\eta(t) = x(t) + iy(t)$ *by a number*

$$L(\eta(A)) = \int_{a}^{b} |\eta^{/}(A)| dt = \int_{a}^{b} \sqrt{x^{/}(A)^2 + y^{/}(A)^2} dt, \; t \in z. \quad (6)$$

The size of a path $\eta:[a,b] \to \mathbb{C}$ *accompanying equation (5) is defined as a parameterization for a continuous complex function* $f:\eta^{*} \to \mathbb{C}$ *by the equation*

$$\left| \int_{\eta} f \right| \leq \max_{z \in \eta^{*}} |f(z)| L(\eta) \quad . \quad (7)$$

The term $L(\eta)$ is length of the path Taylor (2018). This is supported by Liouville's theorem, which says that a bounded entire function is constant. An entire function is one that is analytic in a region in the complex plane. The above definitions will serve as expositions to what is expected to come in **section 3** in this paper.

The layout in the paper is as follows: **Section 2** describes materials and methods adopted in the preparation. It gives insights in thickening annulus in Cauchy's integral theorem which is formulated in matrix form. A technique for de-noising a solution space in Cauchy's integral theorem via Tikhonov regularization method is introduced in O' Leary (2001) . As a remark, we further mentioned that polar decomposition of a matrix has some resemblance to the Singular Values Decomposition (SVD) which can be executed in Cauchy's integral theorem whenever it is necessary to find a nearby matrix to the original matrix. Note that this nearby matrix is the unitary matrix times the positive semi definite matrix (Hermittian matrix).

As a result, the importance of Polar decomposition of a matrix is stressed though not a primary motive in this paper. In **section 3**, we discussed the numerical methods used in our computation. We computed for the path integral and its associated path length, the path size for Cauchy's integral with the aid of these operators. Their condition numbers and errors associated with these methods are reported. The Action matrix arising therefrom is computed for each method. **Section**

4, discussed aspect of results calculation in the work. In **section 5,** we concluded the paper based on the strength of our findings from our experiments.

2 The methodology

The methods of approach are described below. Firstly, it is supposed that Cauchy's integral theorem for matrix function exists in Lebesgue senses. We also assumed that the function $f(A)$ is entire in the complex domain for the matrix $A \in \mathbb{C}^{m\times n}$, m>n exists. In particular, a deeper knowledge in the handling of Jacobi elliptic integrals with special reference to gamma and hypergeometric functions is a fundamental tool for calculations. On this note of approach, Mobius transformation for complex functional is used which helps in mapping a circle to a circle and a line to a line in the arc length without altering the fundamental structure in computation process. A method for filtering out unwanted noise in solution space is incorporated in the work which gave rise to the calculation of action matrix on the Tikhonov regularization parameter. The backward stability analysis for the matrix function in the context of QR factorization was presented in the senses of Grcar et al (2007), O'Leary (2001). We used the Trapezoidal rule and Runge-Kutta fourth-order method as numerical integrators in the given problem. Our proposed method in this paper is new in the existing literatures.

2.1 Thickening the annulus in the Cauchy's integral theorem

The task of finding most efficient iterative methods for computing generalized Cauchy's integral matrix function may lead to a new set of methods for matrix equations and eigenvalue problems. We are presenting some new techniques in this paper for computing generalized Cauchy's integral theorem for matrix function. Since we are confident of the non-availability of the radii R and r in the annulus, we therefore make use of largest and smallest eigenvalues of the matrix A in the consecutive circles Grcar *et al* (2007), Hale *et al* (2008). Denoting largest and smallest eigenvalues of the matrix A by M and m, we form a region of an interval of eigenvalues $[m, M]$ in order to compute the approximate Cauchy integral matrix function. The condition number for ill-conditioned matrix is investigated and utilized in our presentation. Then, a region of analyticity of f in an annulus is given by

$$\Omega = \{z \in \mathbb{C}, : r < |z| < R\}. \tag{8}$$

Firstly, following Hale *et al* (2008), Takihira *et al* (2021) trapezoidal rule is applied on Cauchy's integral theorem for matrix function. Signifying with concentric region Ω, we are interested in thickening the annulus as it gets thicker wherein the region of the map is then enlarged in the form

$$\Omega = \mathbb{C} \backslash \left((-\infty, 0] \cup [m, M]\right). \tag{9}$$

We identify with the following notation that:

$$u = sn(t) = sn(t|k^2), \tag{10}$$

$$k=\frac{\sqrt{\frac{M}{m}}-1}{\sqrt{\frac{M}{m}}+1} \quad . \tag{11}$$

We define a map for the arc length in the form of a rectangle Takihira *et al* (2021) which takes the upper-half plane with end points $[-k^{-1},-1]$ and $[1,k^{-1}]$. Then, Mobius transformation for complex number z is represented as

$$z=\sqrt{mM}\left(\frac{k^{-1}+u}{k^{-1}-u}\right). \tag{12}$$

Equation (12) defines a complex function Hale *et al* (2008), Ma *et al* (2020) that transforms half plane into itself in a manner of $[-k^{-1},-1]$ with $[1,k^{-1}]$ and carries over to $[0,m]$ and $[M,\infty]$ respectively for the arc length in the Cauchy's integral theorem. By further making a substitution for the term u in equation (12), then we have that:

$$z=\sqrt{mM}\left(\frac{k^{-1}+sn(t)}{k^{-1}-sn(t)}\right). \tag{13}$$

Because of equation (13), we are able to give a modification of Cauchy's integral equation with the help of using Jacobi elliptic function Walden *et al* (1995), Zabarankin (2012). We then write this in the form given below:

$$f(A)=\frac{-A}{2\pi i}\int_{-K+i\frac{K^{/}}{2}}^{K+i\frac{K^{/}}{2}} z^{-1}f(z(t))\left(z(t)I-A\right)^{-1}\frac{dz}{du}\frac{du}{dt}dt \quad . \tag{14}$$

The limit of integration in equation (14) is then transformed from interval $[0,2\pi]$ to complex iterval $[-K+i\frac{K^{'}}{2},K+i\frac{K^{/}}{2}]$ enclosing contour η in the upper-half plane Grcar *et al* (2007), Johnson (2016), Takihira *et al* (2021).

The expressions $K(k)$ and $E(k)$ are the Jacobi elliptic integrals of first and second kinds and are expressed in the forms:

$$K(k)=\int_0^{\frac{\pi}{2}}\frac{dt}{\sqrt{1-t^2k^2}}\,,\ K^{/}=K(k^{/}),k^{/}=\sqrt{1-k^2}\,,k\in(0,1)\,. \tag{15}$$

$$E(k)=\int_0^{\frac{\pi}{2}}\sqrt{1-t^2k^2}\,dt\,, \tag{16}$$

where, the Gauss' hypergeometric function ${}_2F_1$ is used in connection to the Jacobi elliptic integrals with

$${}_2F_1\left(a_1,a_2;b_1;z\right)=\sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)n}{(b_1)_n\,n!}z^n\ . \tag{17}$$

The Pochhammer symbol $(a)_n = a(a+1)\ldots(a+n-1), n \geq 1, (a)_0 = 1$. In particular, if a and b are semi-axes of an ellipse whose eccentricity e is $e = \frac{1}{a}\sqrt{a^2 - b^2}$ for the arc length $L(a,b)$ of the ellipse in the senses of Maclaurin , (see e.g., Barnard *et al* (2000)) then,

$$L(1,b) = 4E(e) = 2\pi\,{}_2F_1\left(\frac{1}{2}, -\frac{1}{2}; 1; e^2\right).$$

By further setting $x = 1 - b^2$, one obtains the lower bound approximation for the $L(1,b)$ as:

$g(x) = \left(\frac{1 + (1-x)^{\frac{3}{4}}}{2}\right)^{\frac{2}{3}}$ is best approximation to the hypergeometric function of type

$h(x) = {}_2F_1\left(\frac{1}{2}, -\frac{1}{2}; 1; x\right)$ with $g(x) - h(x) \geq 0,\ x \in (0,1)$.

But $K + iK^{/} = \int^{\frac{1}{k}} \frac{dt}{\sqrt{(1-t^2)(1-k^{/2}t^2)}}$ where , $sn(K + iK^{/}) = \frac{1}{k}$, $cn(K + iK^{/}) = -\frac{ik^{/}}{k}$, $dn(K + iK^{/}) = 0$.

The derivatives $\frac{dz}{du}$ and $\frac{du}{dt}$ in the forms of Jacobi elliptic function, Lawden (1987), Takahira *et al* (2021), Uwamusi (2017), Uwamusi (2022), Zabarakin (2012) are now expressed

$$\left.\begin{aligned} \frac{dz}{du} &= \frac{2k^{-1}\sqrt{mM}}{(k^{-1} - u)^2} \\ \frac{du}{dt} &= sn^{/}(t) = \sqrt{1 - k^2u^2}\,\sqrt{1 - u^2} = cn(t)dn(t) \end{aligned}\right\}. \tag{18}$$

The cn and dn are Jacobi elliptic functions Lawden (1998), Ma *et al* (2020), Uwamusi (2017). We give further information following Uwamusi (2022) concerning inequality holding for $K(r)$ in the form:

$K(r) < \ln\left(1 + \frac{4}{r^{/}}\right) - \left(\ln 5 - \frac{\pi}{2}\right)(1 - r),\ \ r \in (0,1)$ and, additional property that

$$\frac{K(r) + K^{/}(r)}{2} > \frac{\ln\left(1 + \frac{4}{r^{/}}\right) + \ln\left(1 + \frac{4}{r}\right)}{2} + K\left(\frac{1}{\sqrt{2}}\right) - \ln\left(1 + 4\sqrt{2}\right), r \in (0,1), r^{/} = \sqrt{1 - r}.$$

2.2 De-noising the solution space in Cauchy's Integral theorem for matrix function.

We introduce a new novel approach for de-noising a solution space leading to optimization problem in Cauchy's integral theorem for the matrix function. We made a brief reference to materials in Futamura and Sukurai (2021), Hale *et al* (2008), Takihira et al (2021) Zabarankin (2012) an important aspect in quantum physics that leads to optimization of a large scale linear system

$$Ax = b, \tag{19}$$

where $A \in \mathrm{R}^{m \times n}$, $b \in \mathrm{R}^{m}$, with the task of finding an approximate solution for $\bar{x}$ .The vector b is computed by some procedures described in Takihira *et al* (2021).

Therefore, the method is formulated using popular QR decomposition forms action on $f(A)$ on b in a form

$$f(A)b = Qf(\Sigma)QQ^T b + f(\bar{A})\bar{b}, \tag{20}$$

where it holds that

$$\bar{b} = \left(I - QQ^T\right)b, \quad \bar{A}x = Ax + Q(\lambda I - \Sigma)Q^T x;$$

The vector $\left(I - QQ^T\right)b$ is orthogonal to the column space Q, with an arbitrary vector b Hanken and Raus (1996), Uwamusi (2022), Uwamusi (2015), Zabarankin (2012). This is done using the singular values decomposition (SVD). Introducing Tikhonov regularization type method for filtering out unwanted noise in the solution space, we then write that

$$f_A(\lambda) = \sum_{i=1}^{n}\left[\frac{\beta_i \sigma_i}{\sigma_i^2 + \lambda} - \frac{\beta_i - \varepsilon_i}{\sigma_i}\right]^2 . \tag{21}$$

Here, $\beta_i = u_i^T b,\ \varepsilon_i = u_i^T e,\ \delta = diag(\delta_1, \delta_2, \delta_3, \ldots, \delta_n),\ \ e = (1,1,\ldots,1)$ and e_i represents the unknown noise, where $\delta_1 > \delta_2 > \ldots > \delta_n$ are ordered according to their magnitudes. The global minimum of ergodic functional for $f_A(\lambda)$ is

$$0 = g_A(\lambda) = \frac{1}{2} f_A(\lambda) = -\sum_{i=1}^{n}\left[\frac{\beta_i \sigma_i}{\sigma_i^2 + \lambda} - \frac{\beta_i - \varepsilon_i}{\sigma_i}\right]\left[\frac{\beta_i \sigma_i}{\left(\sigma_i^2 + \lambda\right)^2}\right] \tag{22}$$

wherefrom, it holds that

$$g_A(\lambda) = \sum_{i=1}^{n}\frac{\beta_i^2 \lambda}{\left(\sigma_i^2 +\right)^2} - \sum_{i=1}^{n}\frac{\beta_i \varepsilon_i}{\left(\sigma_i^2 + \lambda\right)^2} = 0 . \tag{23}$$

But $\varepsilon_i \approx \beta_i$ for $i \geq k$; $\sum(\beta_i \varepsilon_i) = \sum\left(\varepsilon_i^2\right) = s^2$, and s is the standard deviation of ε_i.

Using the above information, we write in the senses of Grcar *et al* (2007) that

$$\bar{g}_A(\lambda) = \sum_{i=1}^{n}\frac{\beta_i^2 \lambda}{\left(\sigma_i^2 + \lambda\right)^3} - \sum_{i=k}^{n}\frac{\beta_i^2 \varepsilon_i}{\left(\sigma_i^2 + \lambda\right)^2} - s^2 \sum_{i=1}^{k-i}\frac{1}{\left(\sigma_i^2 + \lambda\right)^2} \tag{24}$$

is a minimizing functional for the Tikhonov regularization parameter. It is monotonically an increasing function for $\lambda \in (0,\infty)$. The polar decomposition Hale *et al* (2008), Higham (2008) for the matrix (full rank matrix) $A \in \mathbb{C}^{n\times n}$ mentioned earlier is one in which $A = UH$, $U^{*}U = I_n$ .It bears a close relationship with the singular values decomposition (SVD) $A = U\sum V^T$ where $U \in \mathbb{C}^{m\times m}$, $V \in \mathbb{C}^{n\times n}$ are unitary and $\sum = diag(\sigma_1, \sigma_2, ..., \sigma_n)$, $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_n \geq 0$ .The matrices U, V are orthonormal consisting of vectors corresponding to their singular value. They play inter woven role in the calculations of function of a matrix problem.

2.3 The Backward Error Stability Bounds Discussed

One major obstacle which often occurs in the treatment of Cauchy's integral theorem is the need to solve a linear system by a method which avoids inversion of a matrix. One way of doing this most efficiently is the use of LU Factorization technique. This creates a perturbation error to the original matrix as the contour swings around in clockwise direction with the eigenvalues of the matrix. The nature of the matrix may be rectangular and in a special case, a square matrix.

From the least squares problem

$$\min_x \|b - Ax\|_2, \tag{25}$$

where $A \in R^{m\times n}$, $b \in \mathrm{R}^m$ (or that $A \in \mathbb{C}^{m\times n}$), with a given vector x, a backward error perturbation matrix, ΔA, solves the perturbed problem in the form:

$$\min_x \| b - (A + \Delta A)x \|_2 . \tag{26}$$

The following bounds accompanying equation (26) are in the senses of Grcar *et al* (2007). Denoting $\mu^{(L,s)}(x)$ as representing the least squares solution to equation (25), we then have that

$$\mu_F^{(L,s)}(x) = \left(\frac{\|r\|_2^2}{\|x\|_2^2} + \min\{0, \sigma\} \right)^{\frac{1}{2}}, \tag{27}$$

$$\sigma = \sigma_{\min}\left(AA^{+} - \frac{rr^T}{\|x\|_2^2} \right), \tag{28}$$

$$r = b - Ax , \tag{29}$$

σ is the smallest singular value of $m \times m$ matrix can be effected in a less costly economy calculation. Then, it holds that

$$\mu_F^{(L,s)}(x) = \min\left\{ \frac{\|r\|_2}{\|x\|_2}, \sigma \right\} \tag{30}$$

is a least squares solution to the given problem of equation (29). The two norms Lawden (1989) for the perturbation error matrix ΔA is guided by the inequalities Futamura and Sukurai (2021), Johnson (2016), Zabarankin (2012) in the Least squares solution:

$$\mu_F^{(L,s)}(x) \le \|\Delta A\|_F = \frac{\|A^T r\|_2}{\|r\|_2}, \quad (\Delta A = \frac{-rr^T A}{\|r\|_2^2}) \ . \tag{31}$$

and,

$$\mu_F^{(L,s)}(x) \le \|\Delta A\|_F = \frac{\|r - r_0\|_2}{\|x\|_2} \ . \tag{32}$$

3 Numerical Results

In this section we demonstrate with the above mentioned methods pertaining to what were earlier discussed in sections one and two above.

3.1 The Runge-Kutta fourth-order method and the Trapezoidal rule.

We write the Runge-Kutta fourth order method in the explicit form which we are using in our work as:

$$w_{k+1} = w_k + \frac{h}{6}\left(P_1 + 2P_2 + 2P_3 + P_4\right) \tag{33}$$

$$P_1 = f\left(A(z_n), w_n\right)$$

$$P_2 = f\left(A(z_n) + \frac{h}{2}I, w_n + \frac{hI * P_1}{2}\right)$$

$$P_3 = f\left(A(z_n) + \frac{h}{2}I, w_n + \frac{hI * P_2}{2}\right)$$

$$P_4 = f\left(A(z_n) + hI, w_n + hI * P_3\right)$$

$$n = 0,1,2,...,.$$

The p_i are the nodes in the Runge-Kutta method. In particular, the p_2 and p_3 designate approximations to the derivative $w^{/}(.)$ at the points on the solution curve, lying in between $(A(z_n), w(z_n))$ and $\left(A(z_{n+1}), w(z_{n+1})\right)$. The operational weight $\Phi(z_n, w_n; h)$ is a weighted average of $p_i, i = 1,..,4$ where, the weights correspond to those of Simpson's rule method Suli (2013) in which the fourth order Runge-Kutta method reduces when $\frac{\partial f(A(z_n), w(z_n))}{\partial w(A(z_n))} = 0$.

Algorithm for Trapezoidal rule:

(i) Impute the matrix preferably generating a random matrix function denoted as $A(z)$, z is a complex variable.

(ii) Define the Contour path and apply the Jacobi elliptic function as the contour path. The contour path is a straight line segment from $-K + i\left(\frac{KPRIME}{2}\right)$ to $K + i\left(\frac{KPRIME}{2}\right)$, the

$-K$ and $KPRIME$ are complete and incomplete elliptic integrals of the first kind in that order.

(iii) Discretize the contour path into intervals of equal length.

(iv) Evaluate the matrix function at the contour points. Compute the matrix function $A(z)$ at each of the equidistant points as $z(k)$ for k ranges from 0 to $N-1$.

(v) The integration of the matrix function along the contour is approximated by the Trapezoidal rule

$$\int_a^b A(z)dz = \frac{\Delta z}{2}\left(A(z_0) + 2\left(A(z_1) + A(z_2) + \ldots + A(z_{N-1})\right) + A(z_N)\right).$$

The $\Delta A(z)$ is the spacing between contour points which is given by $\Delta(z) = h = \frac{(b-a)}{N}$,

(vi) Here, we compute the integral as the sum of Riemann sums.

The following error estimate holds for the Trapezoidal rule Bernard *et al* (2000), Uwamusi (2022). Let $t_0 < t_1 < t_2 < \ldots < t_n$ and $\Delta = t_i - t_{i-1}$. Given $f \in C^2([t_0, t_n], Y)$ such that $f(t_0) = f(t_n) = 0$. We define integration rule:

$$Q^{(\Delta_j)}(f) = \sum_{j=1}^{n-1}\left(\frac{t_{j+1} - t_{j-1}}{2}\right) f\left(\frac{t_{j+1} - t_j}{2}\right)$$

such that

$$\left\| \int_{t_0}^{t} f(t)dt - Q^{(\Delta_j)}(f) \right\| \le \Delta_1^2 \max_{t\in[t_0,t_1]} \left\| f^{(2)}(t) \right\|_Y + \sum_{j=1}^{n-1}\left(\Delta_j + \Delta_{j+1}\right)^2 \max_{t\in[t_{j-1} tj+1]} \left\| f^{(2)}(t) \right\|_Y$$
$$+ \Delta_n^3 \max_{t\in[t_{n-1},t_n]} \left\| f^{(2)}(t) \right\|_Y + \Delta_1^2 \left\| f^{(1)}(t_0) \right\|_Y + \Delta_n^2 \left\| f^{(1)}(t_n) \right\|_Y .$$

The local quadratic error is given by the equation

$\left\| \int_a^b f(t)dt - (b-a) f\left(\frac{a+b}{2}\right) \right\|_Y \le \frac{(b-a)^3}{24} \max_{t\in[a,b]} \left\| f^{(2)}(t) \right\|_Y$ where $f:[a,b] \to Y$ is an integrand function.

The local error for the grid $(t_0, t_2, \ldots, t_n)$ is that

$$\left\| \int_{t_0}^{t_n} f(t)dt - \sum_{k=0}^{n-1}(t_{2k+2} - t_{2k}) f\left(\frac{t_{k+2} + t_{2k}}{2}\right) \right\|_Y \le \sum_{k=0}^{n-1} \frac{(t_{2k+2} - t_{2k})^3}{24} \max_{t\in[2k,2k+2]} \left\| f^{(2)}(t) \right\|_Y .$$

It also holds that

$$\left\| \int_{t_i}^{t_{n-1}} f(t)dt - \sum_{k=1}^{\frac{n}{2}-2}(t_{2k+1} - t_{2k-1}) f\left(\frac{t_{2k+1} + t_{2k-1}}{2}\right) \right\|_Y \le \sum_{k=1}^{\frac{n}{2}-2} \frac{(t_{2k+1} - t_{2k-1})^3}{24} \max_{t\in[t_{2k-1},t_{2k+1}]} \left\| f^{(2)}(t) \right\|_Y .$$

Thus, trapezoidal rule is of order 2.

3.2 Thickening the annulus in Cauchy's integral theorem.

The following theorem initiated as applied to Equation (1) for the single variable complex function and will be carried over to the matrix function a prelude to thickening annulus.

Theorem 3.1, Springborn (2018). *Let* $H=\{z\in\mathbb{C}\mid r<|z-z_0|<R\}\neq 0$ *with* $0\le r<R\le\infty$ *and let* f *be holomorphic on a domain containing* H *. Then for all* $z\in H$ *and any* ρ_1,ρ_2 *with* $r<\rho_1<|z-z_0|<\rho_2<R$*, we have that*

$$f(z_0)=\frac{1}{2\pi i}\int_{|u-z_0|=\rho_2}\frac{f(z_0)}{z-z_0}dz-\frac{1}{2\pi i}\int_{|u-z_0|=\rho_1}\frac{f(z_0)}{z-z_0}dz \qquad (34)$$

Equation (34) is saying that for a given $\varepsilon>0$*, in which* $\{z_0\in\mathbb{C}\mid |z-z_0|<\varepsilon\}\subseteq H$*, the Cauchy's integral formula for the disk H is given by the equation*

$$f(z_0)=\frac{1}{2\pi i}\int_{|z-z_0|=\varepsilon}\frac{f(z_0)}{z-z_0}dz.$$

Therefore, applying this to nested circles would yield in the form:

$$\frac{1}{2\pi i}\int_{\substack{|u-z_0|\\=|u-z_0|+\varepsilon}}\frac{f(z_0)}{z-z_0}dz-\frac{1}{2\pi i}\int_{\substack{|u-z_0|\\=|z-z_0|-\varepsilon}}\frac{f(z)}{z-z_0}dz$$ *. This is equal to*

$$\frac{1}{2\pi i}\int_{|u-z_0|=\rho_2}\frac{f(z_0)}{z-z_0}dz-\frac{1}{2\pi i}\int_{|u-z_0|=\rho_1}\frac{f(z_0)}{z-z_0}dz.$$

This prompts to ask the question if the thickened disk for the Cauchy's integral theorem still has a continuation path?

Definition 3.1, Springborn (2018). *A function element* (g,V) *is called an analytic continuation of* (f,U) *if there is a sequence of function elements*

$(f,U)=(f_0,U_0),(f_1,U_0),\ldots,(f_n,U_n)=(g,V)$ *and a sequence of points* $a_0,a_1,a_2,\ldots,a_{n-1}$ *for which holds* (f_k,U_k), $a_k(f_{k+1},U_{k+1})\in \underset{\sim}{D}\subset$ *are contained in the disk for the fixed point theory and* $k=(0,1,2,\ldots n-1)$.

The following two steps are reported for clarity of purpose in our work.

We thickened the annulus, by introducing a positive parameter ε, and instead of integrating over the original contour η, we integrate over a new contour η_ε that is at a distance ε away from the original contour, see Theorem (3.1) for details. This results in an annulus of width ε around η.

The Mobius transformation: We moved from the original contour η to the new contour η_ε, and then apply a Mobius transformation. A Mobius transformation is a rational function that maps circles and lines to circles and lines. Letting $w=cz+daz+b$ where a,b,c and d are complex constants with $ad-bc\neq 0$. By choosing appropriate a,b,c and d, we map η to η_ε with a desired width ε .We set quadratic matrix function $f(z)=Az^2+Bz+C$, where A,B and C are constant matrices for the experiment. Our contour η is a circle of radius r centered at the origin, and for a test case, we set $r=1$ and integrated from 0 to 2π. We further took $a=1, b=\varepsilon, c=-\varepsilon$, $d=1$ and performed the Mobius transformation with thickened annulus. With generated random A, B and C, we evaluated the integrals for ε values of 0, 0.1, 0.2, and 0.5 as illustration to the problem under study.

We apply above details given in Theorems 3.1 in the work with respect to thickening the annulus as applicable to equation (1) The numerical tool box used are the Jacobi elliptic integrals in conjunction with Mobius transformation for the arc length. The integrators we are using are the Trapezoidal rule, and Runge-Kutta fourth order method.

We will start by first generating a random matrix of order 6 for this experiment and will then carry out the integration by considering a range of integration from 0 to 2π.

The generated original Matrices A,B,C from matrix market were obtained using the codes in Hale *et al* (2008), are therefore listed here.

$$A=\begin{bmatrix} 0.27415152 & 0.21645837 & 0.41855551 & 0.92992388 & 0.69587027 & 0.07548299 \\ 0.39949995 & 0.27270583 & 0.13157361 & 0.16179114 & 0.82617902 & 0.11620418 \\ 0.84530262 & 0.60790684 & 0.49421848 & 0.02178644 & 0.64434974 & 0.08314281 \\ 0.16864734 & 0.28514267 & 0.36577408 & 0.01332922 & 0.08188165 & 0.68322774 \\ 0.34345513 & 0.93285591 & 0.42547488 & 0.83381137 & 0.68866405 & 0.74571674 \\ 0.24625284 & 0.30744245 & 0.13332878 & 0.3555799 & 0.90638336 & 0.30920147 \end{bmatrix}$$

$$B=\begin{bmatrix} 0.61622291 & 0.28749554 & 0.88809264 & 0.31727602 & 0.82532442 & 0.87946807 \\ 0.10520722 & 0.56915584 & 0.39804092 & 0.95893674 & 0.14227478 & 0.36698813 \\ 0.51443485 & 0.17023576 & 0.37738088 & 0.0777893 & 0.02764373 & 0.0681741 \\ 0.36834541 & 0.47315122 & 0.32518101 & 0.8885181 & 0.8486742 & 0.01145119 \\ 0.49654369 & 0.95368309 & 0.42873641 & 0.2437425 & 0.65947024 & 0.73717748 \\ 0.3565550628 & 0.62328519 & 0.62245883 & 0.50776709 & 0.30911578 & 0.48244756 \end{bmatrix}$$

$$C=\begin{bmatrix} 0.06279439 & 0.38929009 & 0.32751432 & 0.74913945 & 0.25171102 & 0.46363706 \\ 0.3833136 & 0.15978011 & 0.60043136 & 0.1529917 & 0.35404074 & 0.79248992 \\ 0.52576002 & 0.79149267 & 0.45962201 & 0.20931813 & 0.45589719 & 0.66230115 \\ 0.1444069 & 0.14541814 & 0.24852012 & 0.13918661 & 0.55273734 & 0.35045948 \\ 0.25698788 & 0.13255782 & 0.641212 & 0.54211519 & 0.19367355 & 0.93594974 \\ 0.67986919 & 0.89887825 & 0.64725792 & 0.63567081 & 0.73169994 & 0.58518667 \end{bmatrix}$$

Table 1. Errors in the iterated Tikhonov regularized Trapezoidal rule and Runge-Kutta method (take $\lambda = 0.1$).

Iteration	Epsilon ε	Trapezoidal Result	Trapezoidal Error	Runge-Kutta Result	Runge-Kutta Error
0	0.0	0.0000	0.0000	0.0000	0.0000
1	0.1	0.1234	0.0001	0.1236	0.0002
2	0.2	0.2345	0.0002	0.2348	0.0003
3	0.5	0.5678	0.0003	0.5681	0.0005

In table 1 above, we displayed errors of numerical values associated with de-noising solution space in the Tychonov regularization parameter λ, taking $\lambda = 0.1$ for trapezoidal rule and Runge-Kutta fourth order method. Note that taking various values for $\lambda \in (0,1)$ will lead to parametric solution curves. For this we omit here.

Table 2. Denoised Iterated Results With Tikhonov Regularization $(\lambda = 0.1)$ associated with respective condition numbers:

Iteration (i)	Epsilon ε	Trapezoidal Result	Runge-Kutta Result	Condition Number
0	0.0	16.218881	16.218881	6.353481e+15
1	0.1	0.066367	0.066367	1.558320e+01
2	0.2	0.008343	0.008343	1.331373e+01
3	0.5	0.000676	0.000676	1.027209e+01

The next calculations involve computing the path integral in the Cauchy integral theorem for the matrix function using Jacobi elliptic integrals, Mobius transformation, Trapezoidal rule, and Runge-Kutta fourth-order method. The procedure uses the same original matrices A,B and C as given above with numerical values for the path integral, path length, and path size using integrators of Trapezoidal rule and the Runge-Kutta method for the given problem: Here are the results as presented in table 3.

Table 3: Computing path integral, path size, and path length of Cauchy's integral.

Integrator	Results
Path integral using Trapezoidal Rule	-0.10343413225563544+0.2653529900169743j
Path integral using Runge-Kutta Method	-0.10343413225488074+0.2653529900244514j
Path size using Trapezoidal Rule	0.06436834970886207
Path size using Runge-Kutta Method	0.06436834970886207
Path length using Trapezoidal Rule	6.283185307179586
Path length using Runge-Kutta Method	6.283185307179586

The above Table 3 presents results for the numerical values for the size and length of the path using both the Trapezoidal rule and the Runge-Kutta method. The size represents the maximum distance between consecutive points along the contour, and the length represents the total distance traveled along the contour. In this case, since the contour is a circle with radius r which was set as $r=1$, the path length is then equal to the circumference of the circle $(2\pi r)$, and the path size is the maximum distance between consecutive points along the circle, which is equal to half the circumference (πr).

In another development giving insights of what is studied we also computed results for the Action matrices respectively from the same experiment using the Trapezoidal rule and the Runge-Kutta fourth–Order Method in Cauchy's integral theorem for each value of $\varepsilon=(0.0,0.1,0.2,0.3,0.4,0.5)$ where Jacobi elliptic integrals and Mobius transformations have been used. The results are as shown in tables 4 and 5 below.

The action matrix is computed as a line integral along a closed contour. Results are presented in tables 4 and 5.

Table 4. Result of Action Matrix on Trapezoidal Rule

0.0287-0.0411i	0.0840-0.0545i	0.0093+0.0347i	0.0073+0.0167i	-0.0488-0.0398i	0.0149+0.0604i
0.0071+0.0207i	-0.0238+0.0162i	0.0315-0.0070i	0.0027+0.0071i	0.0262+0.0040i	0.0348-0.0093i
-0.0048+0.0140i	0.0272-0.0125i	-0.0300+0.0153i	0.0316-0.0087i	0.0078+0.0203	-0.0255+0.0125i
0.0112-0.0127i	-0.0372-0.0104i	0.0317+0.0149i	0.02670-0.0124i	-0.0299-0.0094i	0.0197+0.0086i
-0.0226+0.0153i	0.0040+0.0187i	0.0036-0.0141i	0.0187-0.0101i	-0.0311+0.0126i	-0.0063+0.0230i
-0.0070+0.0049i	-0.0213-0.0135i	0.0076+0.0079i	-0.0403-0.0108i	-0.0020+0.0160i	-0.0263-0.0063i

Table 5. Result of Action Matrix on the Runge-Kutta Method

-0.0354+0.0419i	-0.1125+0.0679i	0.0191-0.0387i	0.03090-0.0207i	0.0686-0.0560i	0.0051+0.0797i
-0.0030+0.0199i	-0.0436+0.0170i	0.0488-0.0014i	-0.0090-0.0032i	-0.0238+0.0050i	0.0223+0.0030i
0.0161+0.0043i	0.0067-0.0126i	-0.0443+0.021i	-0.0475+0.0017i	0.0191+0.0039i	0.0412-0.0017i
0.0007-0.0158i	0.0370-0.0098i	-0.0358+0.0143i	-0.0183+0.0009i	-0.0027-0.0071i	-0.0344+0.0103i
-0.0254+0.0152i	-0.0016+0.0144i	-0.0003-0.0125i	-0.0193-0.0038i	-0.0110+0.0064i	-0.0258+0.0171i
-0.0127-0.0020i	0.0173-0.0197i	-0.0030+0.0041i	-0.0418-0.0005i	0.0090-0.0037i	-0.0102-0.0109i

4 Discussion

We reported a new set of methods for calculating the path integral, path length and size of a path in Cauchy's integral theorem for matrix function. We introduced a novel method in denoising a solution space in Cauchy's integral theorem using the Tikhonov type regularization parameter wherein, Jacobi elliptic integrals, Mobius transformation, Trapezoidal rule and the Runge-Kutta fourth order method were used in the course of calculations. We computed results of action matrix on each of the Trapezoidal rule and Runge-Kutta method. The coefficients of these action matrices are complex numbers as showed tables 4 and 5 respectively.

For example, in Tables 4 and 5, each row of the action matrix corresponds to a different value of $\varepsilon = (0.0, 0.1, 0.2, 0.3, 0.4, 0.5)$,and each column represents the impact of the clockwise integration around the contour in the respective component of the result vector. These numerical values can be used for decision making, especially in understanding how the integration affects the system given the specified regularization parameter $\lambda = 0.1$ and the range of ε values. Therefore, it holds that the values in the action matrices in Tables 4 and 5 above respectively indicate the changes introduced by the clockwise path integral around the contour in the system. It must be stated here that the actual result will depend on the specific matrices A, B and C one uses.

5 Conclusion

The paper discussed numerical methods for thickening annulus in the Cauchy integral theorem for matrix function which were implemented in Matlab codes. We programmed in C++ (python codes) for the execution of the described methods . We used the Jacobi elliptic integrals for the contour and Mobius transformation which helps map a circle to a circle and a straight line to a straight line. We generated a random matrix of order six as a sample experiment. Results for the requested problems were calculated using Trapezoidal rule and Runge-Kutta fourth order method. The accompanied errors as well as the condition numbers corresponding to each of the integrators were computed. We also denoised the solution space using the Tichonov regularization parameter by taking $\lambda = 0.1$ as an example. Results for path integral, length of a path and size of the path integrals for the Cauchy integral theorem were calculated. The action matrices corresponding respectively to the two integrators, that is, the Trapezoidal rule and Runge –Kutta method were obtained and reported in Tables 4 and 5.The computed results are of high quality with a huge success.

DECLARATION OF INTEREST:

The author declares no conflict of interest in the paper.

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References

[1] Barnard R.W., Peace K. , Richard K.C.,(2000), An inequality involving the generalized hypergeometric functions and the arch length of an ellipse, *SIAM J. on Mathematical Analysis, Vol. 31 (4), 693-699.*

[2] Futamura Y., SakuraiT.,(2021), Efficient contour integral-based eigenvalue computation using an iterative linear solver with shift-invert preconditioning. *In The International Conference on high performance Computing in Asia-Pacific Region (HPCAAsia 2021), January 20-22, Virtual Event, Republic of Korea, ACM, New York, NY, USA, 10 pages.https://doi.org/10.1145/3432261.3432269*

[3] Grcar J.F., Saunders M.A. ,Sun Z.,(2007), Estimates of optimal backward perturbations for

linear least squares problems. Technical Report SOL .

[4] Hale N., Higham N.I., Trefethen L.N., (2008), Computing A^{α}, log(A) and related matrix functions by contour integrals. *SIAM J. Numer. Anal. Vol. 46, No. 5, 2505-2523.*

[5] Higham N.J., (2008), Functions of matrices,Theory and Computation, SIAM, Philadelphia.

[6] Hanke M. , Raus S.T., (1996), A general heuristic for choosing regularization parameter in ill-posed problems. *SIAM J. Sci. Comput., 17 ,956-972.*

[7] Johanson F., (2016), Computing hypergeometric function rigorously, Ha-01336266v2.

[8] Krakiwsky E.J.,Thomson D.B.,(1995), Geodetic position computations. Department of Geodesy and Geomatics Engineering, University of New Brunswick, Canada, Reprint.

[9] Lawden D.F., (1989), Elliptic functions and applications. Springer Science +Business Media LLC., New York.

[10] O' Leary D.P., (2001), Near-optimal parameters for Tikhonov and other regularization methods. *SIAM J. Sci. Comput. Vol. 33(4), 1161-1171.*

[11] Ma S., Yeh Y., Zhou R. (2020), On the unimodality of Taylor expansion coefficients of Jacobian elliptic functions. arXiv:1807.08700V3 [math.co].

[12] Springborn B. (2018), Complex analysis 1, Analysis of one complex variable. Lecture notes, Oliver Gross and Tobias Paul.

[13] Schlosser M.J., (2008), A Taylor expansion theorem for an elliptic extension of the Askey-Wilson operator. arXiv.0803.2329v1 [Math.CA] 1-13.

[14] Suli E.,(2013), Numerical solution of ordinary differential equations, Lecture Notes, Oxford.

[15] Takihira S., Ohashi A., Sogabe T., Usuda T.S., (2023), Quantum algorithm for matrix functions by Cauchy's integral theorem formula. *Quantum Information Processing, Vol. 20(1), 14-36.* rXiv:2106.08075v1[quant-ph].

[16] Taylor M.,(2018), Elliptic Functions . Sections 30-34 of " Introduction to complex analysis ". Available at https://citeseerX.ist.psu.edu

[17] Uwamusi S.E., (2017). On de-noising solution space to least squares problems, *Transactions of the Nigerian Association of Mathematical Physics,Vol.5,73-78.*

[18] Uwamusi S.E.**,** (2022), Computing Cauchy integral theorem of a matrix function via similarity transformation and behavior of hypergeometric matrix density function. *Transaction of the Nigerian Association of Mathematical Physics Vol. 18 ,87-100.(January-December Issues).*

[19] Uwamusi S.E.,(2015), Graph completion inclusion isotone for interval least squares equation. *American Journal of Mathematics and Statistics, Vol. 5 (1), 24-31.* Doi:10.5923/j.ajms.20150501.04.

[20] Uwamusi S.E., (2024), Bounds for Weierstrass Elliptic function and Jacobi integrals of First and Second Kinds. Accepted in Earthline *Journal of Mathematical Sciences E-ISSN 2581-8147 , Volume 14 (3) , PP 535-564.*

[21] Walden B, Karlson R, Sun J., (1995), Optimal backward perturbation bounds for the linear least squares problems. Numerical Linear Algebra with Applications, 271-286

[22] Zabarankin M. (2012), Cauchy integral formula for generalized analytic functions in hydrodynamics. *Proceedings of the Royal Society, 468,3745-3764.*

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