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Falkner Type Hybrid Block Method for Solving Second-order IVPs in ODEs

R. I. OKUONGHAE* and I. B. AIHIE

Department of Mathematics, Faculty of Physical Sciences, University of Benin, Nigeria

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Abstract

This study uses Falkner-type hybrid block methods to address the numerical solution of second-order ordinary differential equations (ODEs). The primary challenge with traditional approaches is their limited efficiency and accuracy in handling stiff ODEs and complex boundary conditions. A novel hybrid block method that integrates the Falkner framework with fixed step-size techniques is proposed to overcome these limitations. The coefficients of the new schemes are obtained via Taylor's series expansion. The transformation of the derived methods to the block scheme promotes easy implementation and enhances convergence. The hybrid scheme is implemented, and its performance is validated through a series of benchmark problems, comparing the numerical solution of the proposed scheme with the exact solution of ODEs problems. The efficacy of this method is demonstrated with two numerical experiments, elucidating its stability and convergence properties. The results show the computational efficacy of the new scheme, which shows that the proposed scheme is a novel numerical tool for solving second-order ODEs with implications for various application areas in engineering and applied mathematics.

Keywords: *Falkner hybrid method, hybrid block method, second-order initial value problems, oscillatory problems, stiff problems.*

1. INTRODUCTION

Solving ordinary differential equations (ODEs) is a fundamental task in various fields of science and engineering, encompassing dynamics, control systems, and mathematical modeling. Analyzing these equations is crucial for understanding the dynamics of systems modeled by them.

One common approach is the method of characteristic equations, often applied to linear ODEs with constant coefficients. This method involves finding solutions in

*Corresponding author e-mail: iyekeoretin.okuonghae@uniben.edu

the form of exponentials, allowing for straightforward integration of the terms. For non-homogeneous equations, the method of undetermined coefficients and parameter variation can be employed to find particular solutions that complement the homogeneous solution.

Additionally, techniques such as the Laplace transform can facilitate the solution of initial value problems, transforming the ODE into an algebraic equation. For second-order linear ODEs, series solutions can also be utilized, particularly when dealing with variable coefficients or when singular points are present. Several analytical methods (the Adomian decomposition method in Zeidan et al. (2020), the homotopy type method in Turq (2020), the Perturbation Iteration method in Singh and Reddy (2020), Pakdemirli and Aksoy (2024)) exist for solving second-order ODEs, each suited for different types of equations based on their characteristics.

In situations where the analytical methods fail, the numerical formula for second-order ODEs comes in handy. Interestingly, the second-order initial value problems (IVPs),

$$\begin{cases} y''(x) = f\left(x, y(x), y'(x)\right), \\ y(x_0) = y_0, \quad y'(0) = y'_0, \\ x \in \left[x_0, x_n\right], \end{cases} \tag{1}$$

are particularly significant due to their rich applications, ranging from mechanical vibrations to fluid dynamics. Traditional numerical methods, such as Euler's method and the Runge-Kutta family, (see Fatunla, (1988), Lambert, (1973), Lambert, (1991), and Butcher (2008)) provide established approaches for solving these equations but often come with limitations regarding accuracy and stability, especially when faced with stiff systems or high-order dynamics.

In recent years, there has been a growing interest in formulating more sophisticated numerical methods that can enhance solution accuracy and computational efficiency.

The rest of this paper is organized as follows: Section 2 deals with the general formulation, order conditions, the block format, and the stability analysis of the proposed Falkner-type hybrid linear multistep method (FHLMM). In Section 3, numerical experiments are carried out, and the numerical results show the robustness of the new methods. Section 4 is concerned with conclusions and discussions.

2. MATERIALS AND METHOD

This paper introduces a new hybrid block method with two hybrid points specifically designed for the numerical solution of the second-order initial value problems (1). The proposed methods employ a predictive structural framework defined by the equations:

$$\begin{cases} y_{n+c} = a_0 y_{n+1} + a_1 h y'_{n+1} + h^2\left(b_0 f_n + b_1\left(f_{n+1} + f_{n-v-1}\right) + b_2\left(f_{n+2} + f_{n-v+1}\right)\right), \\ y'_{n+c} = \sigma_0 y'_{n+1} + h\left(g_0 f_n + g_1\left(f_{n+1} + f_{n-v-1}\right) + g_2\left(f_{n+2} + f_{n-v+1}\right)\right), \\ c = \left(c_0, c_1, \cdots, c_k\right),\ y'' = f_x + f f_y. \end{cases} \tag{2}$$

where, as usual, the y_{n+c} and y'_{n+c} are the approximate solution points to the exact solutions at $y(x)$, and $y'(x)$ respectively. Here, h represents the step size, and the coefficients a_0, a_1, σ_0, b_0, b_1, b_2, g_0, g_1, and g_2 are meticulously chosen to optimize the accuracy and stability of the numerical method in (2). By incorporating information from multiple previous time steps and their derivatives, these methods achieve higher-order accuracy than many classical techniques, effectively reducing numerical error while maintaining computational efficiency (See Okuonghae and Ozobokeme, 2024). Vector c contains a set of data or points on the real line on the x-axis. At any point where the value of the off-step point v coincides with a point in the vector c, a hybrid method is formed. The off-step point, v in (2), lies between $-\infty$ and 2. The value of v is chosen arbitrarily to achieve the desirable stability properties. The structural formulation of the method in (2) is different from the methods in Jator (2010), Ramos et al. (2016), and Ramos et al. (2017).

2.1 The Derivation of the Hybrid Method

This subsection explains how the implicit hybrid method in (2) is derived. Expanding the methods in (2) via Taylor's series yields the following order conditions for the first method in (2):

$$\begin{gathered} -(-1 + a_0) = 0, \qquad -(a_0 + a_1 - c) = 0, \\ -\tfrac{1}{2}\left(\left(a_0 + 2\left(a_1 + b_0 + 2(b_1 + b_2)\right) - c^2\right)\right) = 0, \\ \frac{1}{6}(-a_0 - 3a_1 + 6vb_1 + 6(-3 + v)b_2 + c^3) = 0, \\ -\tfrac{1}{24}\left(\left(a_0 + 4\left(a_1 + 3\left(2 + v(2 + v)\right)b_1 + 3(5 + (-2 + v)v)b_2\right) - c^4\right)\right) = 0. \end{gathered} \tag{3}$$

In like manner, the order conditions for the second method in (2) are as follows,

$$\begin{gathered} -(-1 + \sigma_0) = 0, -(a_1 + g_0 + 2(g_1 + g_2) - c) = 0, \\ \tfrac{1}{2}(-a_1 + 2vg_1 + 2(-3 + v)g_2 + c^2) = 0, \\ \tfrac{1}{6}((\sigma_0 + 3(2 + v(2 + v))g_1 + 3(5 + (-2 + v)v)g_2 - c_0^3)) = 0. \end{gathered} \tag{4}$$

Solving the above system in (3) and (4) for the values of the coefficients a_0, a_1, σ_0, b_0, b_1, b_2, g_0, g_1, and g_2, respectively, and substituting the arising results into (2) gives

$$y_{n+c} = a_0 y_{n+1} + a_1 h y'_{n+1} + h^2(b_0 f_n + b_1(f_{n+1} + f_{n-v-1}) + b_2(f_{n+2} + f_{n-v+1})), \tag{5}$$

where

$$a_0 = 1, \qquad a_1 = -1 + c,$$

$$b_0 = \frac{-(21 - 42c + 18c^2 + 6c^3 - 3c^4 + 11v - 30cv + 27c^2v - 8c^3v - 3v^2 + 6cv^2 - 3c^2v^2)}{6(-6 - 9v + v^2)},$$

$$b_1 = \frac{-(-11 + 18c - 10c^3 + 3c^4 + 5v - 8cv + 4c^3v - c^4v - 4v^2 + 6cv^2 - 2c^3v^2)}{12(-6 - 9v + v^2)},$$

$$b_2 = \frac{-(8 - 12c + 4c^3 + 11v - 16cv + 4c^3v + c^4v + 4v^2 - 6cv^2 + 2c^3v^2)}{12(-6 - 9v + v^2)}.$$

Similarly, the derivative method gives,

$$y'_{n+c} = \sigma_0 y'_{n+1} + h(g_0 f_n + g_1(f_{n+1} + f_{n-v-1}) + g_2(f_{n+2} + f_{n-v+1})), \quad (6)$$

where

$$\sigma_0 = 1, \quad g_0 = -\frac{-7 + 6c + 3c^2 - 2c^3 - 5v + 9cv - 4c^2v + v^2 - cv^2}{-6 - 9v + v^2},$$

$$g_1 = -\frac{9 - 15c^2 + 6c^3 - 4v + 6c^2v - 2c^3v + 3v^2 - 3c^2v^2}{6(-6 - 9v + v^2)},$$

$$g_2 = -\frac{-6 + 6c^2 - 8v + 6c^2v + 2c^3v - 3v^2 + 3c^2v^2}{6(-6 - 9v + v^2)}.$$

Fixing $v = \frac{1}{2}$ and $c = (-\frac{3}{2}, 0, \frac{1}{2}, 2)^T$ in (5) and (6) gives the following formulas,

$$y_{n-\frac{3}{2}} = y_{n+1} - \frac{5}{2}hy'_{n+1} + h^2\left(\frac{625}{246}f_n + \frac{275}{3936}\left(f_{n-\frac{3}{2}} + f_{n+1}\right) + \frac{875}{3936}\left(f_{n+\frac{1}{2}} + f_{n+2}\right)\right),$$

$$y'_{n-\frac{3}{2}} = y'_{n+1} + h\left(-\frac{125}{82}f_n - \frac{605}{984}\left(f_{n-\frac{3}{2}} + f_{n+1}\right) + \frac{125}{984}\left(f_{n+\frac{1}{2}} + f_{n+2}\right)\right),$$

$$y_n = y_{n+1} - hy'_{n+1} + h^2\left(\frac{103}{246}f_n - \frac{19}{246}\left(f_{n-\frac{3}{2}} + f_{n+1}\right) + \frac{29}{246}\left(f_{n+\frac{1}{2}} + f_{n+2}\right)\right),$$

$$y'_n = y'_{n+1} + h\left(-\frac{37}{41}f_n + \frac{31}{246}\left(f_{n-\frac{3}{2}} + f_{n+1}\right) - \frac{43}{246}\left(f_{n+\frac{1}{2}} + f_{n+2}\right)\right), \quad (7)$$

$$y_{n+\frac{1}{2}} = y_{n+1} - \frac{1}{2}hy'_{n+1} + h^2\left(\frac{23}{246}f_n - \frac{85}{3936}\left(f_{n-\frac{3}{2}} + f_{n+1}\right) + \frac{49}{1312}\left(f_{n+\frac{1}{2}} + f_{n+2}\right)\right),$$

$$y'_{\frac{1}{2}+n} = y'_{n+1} + h\left(-\frac{33}{82}f_n + \frac{83}{984}\left(f_{n-\frac{3}{2}} + f_{n+1}\right) - \frac{131}{984}\left(f_{n+\frac{1}{2}} + f_{n+2}\right)\right),$$

$$y_{n+2} = y_{n+1} + hy'_{n+1} + h^2\left(\frac{23}{246}f_n - \frac{13}{246}\left(f_{n-\frac{3}{2}} + f_{n+1}\right) + \frac{21}{82}\left(f_{n+\frac{1}{2}} + f_{n+2}\right)\right),$$

$$y'_{n+2} = y'_{n+1} + h\left(-\frac{3}{41}f_n - \frac{13}{246}\left(f_{n-\frac{3}{2}} + f_{n+1}\right) + \frac{145}{246}\left(f_{n+\frac{1}{2}} + f_{n+2}\right)\right).$$

2.2 The Block Format of the Hybrid Method in (2)

In the spirit of Ramos et al. (2016), the block method in (7) is define as,

$$AY_n = hBY_n{}' + h^2CF_n \tag{8}$$

where A, B, C are matrices of coefficients of dimensions of 8 by 5, and

$$Y_n = (y_{n-v-1}, y_n, y_{n-v+1}, y_{n+1}, y_{n+2})^T,$$
$$Y_n{}' = (y'_{n-v-1}, y_n{}', y'_{n-v+1}, y'_{n+1}, y'_{n+2})^T,$$
$$F_n = (f_{n-v-1}, f_n, f_{n-v+1}, f_{n+1}, f_{n+2})^T,$$

with $v = \frac{1}{2}$. The picture of (7) in (8) format is,

$$A = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \qquad B = \begin{pmatrix} 0 & 0 & 0 & \frac{-5}{2} & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{-1}{2} & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix};$$

$$C = \begin{pmatrix} \frac{275}{3936} & \frac{625}{246} & \frac{875}{3936} & \frac{275}{3936} & \frac{875}{3936} \\ \frac{-605}{984} & \frac{-125}{82} & \frac{125}{984} & \frac{-605}{984} & \frac{125}{984} \\ \frac{-19}{246} & \frac{103}{246} & \frac{29}{246} & \frac{-19}{246} & \frac{29}{246} \\ \frac{31}{246} & \frac{-37}{41} & \frac{-43}{246} & \frac{31}{246} & \frac{-43}{246} \\ \frac{-85}{3936} & \frac{23}{246} & \frac{49}{1312} & \frac{-85}{3936} & \frac{49}{1312} \\ \frac{83}{984} & \frac{-33}{82} & \frac{-131}{984} & \frac{83}{984} & \frac{-131}{984} \\ \frac{-13}{246} & \frac{23}{246} & \frac{21}{82} & \frac{-13}{246} & \frac{21}{82} \\ \frac{-13}{246} & \frac{-3}{41} & \frac{145}{246} & \frac{-3}{41} & \frac{145}{246} \end{pmatrix}.$$

As in Lambert (1991), if $\psi(x)$ is an arbitrary and sufficiently differentiable function, then the linear difference operator $\mho$ associated with the block method in (8) is,

$$\mho[\psi(x); h] = \psi(x_n + ch) - \bar{a}_0\psi(x_n + h) - h\bar{a}_1\psi'(x_n + h) - h^2(\bar{\Omega}_0\psi''(x_n)$$
$$+\bar{\Omega}_1(\psi''(x_n + h) + \psi''(x_n - (v + 1)h)) + \bar{\Omega}_2(\psi''(x_n + 2h) + \psi''(x_n - (v - 1)h))), \tag{9}$$

and

$$\mho'[\psi(x);h] = \psi'(x_n + ch) - \bar{\sigma}_0\psi'(x_n + h) - h(\bar{\Phi}_0\psi''(x_n)$$
$$+\bar{\Phi}_1(\psi''(x_n + h) + \psi''(x_n - (v+1)h)) + \bar{\Phi}_2(\psi''(x_n + 2h) + \psi''(x_n - (v-1)h))). \tag{10}$$

where $\bar{a}_0$, $\bar{a}_1$, $\bar{\Omega}_0$, $\bar{\Omega}_1$, $\bar{\Omega}_2$, σ_0, $\bar{\Phi}_0$, $\bar{\Phi}_1$, and $\bar{\Phi}_2$ are the vector columns of the matrices A, B, C. By Taylor's series expansion of $\psi(x_n + ch)$, $\psi'(x_n + ch)$, $\psi(x_n + h)$, $\psi'(x_n + h)$, $\psi''(x_n)$, $\psi''(x_n + h)$, $\psi''(x_n - (v+1)h)$ $\psi''(x_n - (v-1)h))$, and $\psi''(x_n + 2h)$ in (9) and (10) about x_n respectively we obtain

$$\mho[\psi(x);h] = \bar{C}_0\, y(x_n) + \bar{C}_1 h\, y'(x_n) + \bar{C}_2\, h^2 y''(x_n) + \ldots + \bar{C}_q\, h^q y^{(q)}(x_n) + \ldots$$

The order p of the difference operator $\mho[\psi(x);h]$ is a unique integer p such that $\bar{C}_q$, $q = 0(1)p+1$, $\bar{C}_{p+2} \neq 0$. The $\bar{C}_{p+2}$ is the error constant of the method in (7) and is given by

$$C_5 = \left(\frac{-11375}{47232}, \frac{-3875}{15744}, \frac{-4631}{29520}, \frac{241}{984}, \frac{-10949}{236160}, \frac{2749}{15744}, \frac{-4459}{29520}, \frac{-191}{984}\right)^T.$$

The order of the method in (8) is $p = 3$. In the spirit of Fatunla (1991), the following definition is given.

Definition 1 (cf. Fatunla (1991)) A block method (8) is zero-stable as $h \to 0$, and if the roots of the first characteristic polynomial have modulus less than or equal to one and those of modulus one do not have multiplicity greater than 2, i.e. the roots of

$$\rho(w) = \det\left(\sum_{i=0}^{k} A^{(i)} w^{k-i}\right) = 0, \quad A^{(0)} = -I, \tag{11}$$

satisfy $|w_j| \le 1$ and for those roots with $|w_j| = 1$, the multiplicity does not exceed 2.

In the spirit of Fatunla (1991), the first characteristic matrices of the method in (7) is obtained as,

$$A^{(0)}Y_n - A^{(1)}Y_{n-1} = 0,$$

where

$$A^{(0)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};\ A^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix};\ Y_n = \begin{pmatrix} y_{n-3/2} \\ y_n \\ y_{n+1/2} \\ y_{n+2} \end{pmatrix};\ Y_{n-1} = \begin{pmatrix} y_{n-5/2} \\ y_{n-1} \\ y_{n-1/2} \\ y_{n+1} \end{pmatrix}.$$

From (11), the first characteristic polynomial of (8) is

$$\rho(w) = \det\left(-\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} w^2 + \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mathrm{w} \right) = -w^7(1-w) = 0.$$

This result obtained from the zero-stability analysis shows that the block method in (7) is zero-stable, consistent, and convergent since the roots of the first characteristics polynomial satisfied $|\, w_j| \leq 1$, and the order of the new block is greater than 1.

2.3 The Stability Analysis of the Block Method in (2)

This subsection will analyze the stability of the block method applied to the generalized second-order ODEs test scalar problem of the form,

$$y''(x) = -2\mu y'(x) - \mu^2 y(x). \tag{12}$$

Stability is crucial in numerical analysis as it dictates how errors propagate through computations and whether the numerical solutions converge to the true solution. The definition of absolute stability follows immediately.

Definition 2 The block method in (8) is said to be stable if the interval of absolute stability is (0, Þ), where Þ is a real number.

Applying the formula in (8) to (12) yields a system of eight equations involving five distinct derivative terms $y'_{n-\frac{3}{2}}$, y'_n, $y'_{n+\frac{1}{2}}$, y'_{n+1}, y'_{n+2}. To simplify the analysis, we employed MATHEMATICA software to eliminate these derivative terms from the equations, resulting in three stability polynomials expressed solely in terms of $y_{n-\frac{3}{2}}$, y_n, $y_{n+\frac{1}{2}}$, y_{n+1}, y_{n+2}. As an example, one of the derived stability polynomials is,

$$\pi(w,z) = 85w^2z^2 + w^e(-23256 + 21494z - 4855z^2) + 20(1056 + 6167z - 2166z^2) + 5w^d(14808 - 28542z + 17z^2) - w(71904 + 2124z + 4855z^2), \tag{12}$$

where $z = \mu h$, $d = \frac{1}{2}$, $e = \frac{-3}{2}$. Plotting the absolute values of the roots of the stability polynomial in (13) in the boundary locus sense yields the method's stability region to be (-0.2869, 0); see Figure 1.

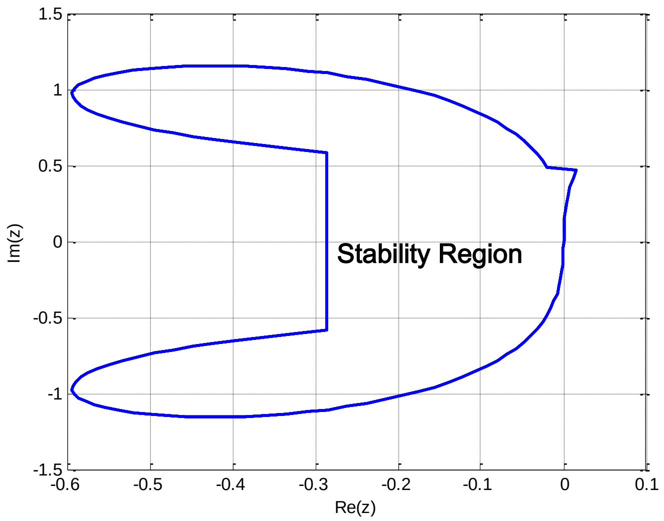

Figure 1: *The stability region (the interior part) of the methods in (8)*

3. RESULT AND DISCUSSION

This section presents the results of applying the Falkner-type hybrid block method to two test problems, demonstrating its capabilities and effectiveness. The performance metrics such as accuracy and computational resources utilized will be discussed. Furthermore, a critical analysis of the advantages of the Falkner-type hybrid block method will be highlighted to show its robustness and versatility in handling a diverse range of ODE problems.

3.1 Numerical Experiment and Results

In this subsection, detailed results will be presented, illustrating the method's performance across various parameters and conditions. The block method in (11) is implicit. Therefore the non-linearity arising from the method when applied to the test problems is resolved using a modified Newton Raphson scheme using the explicit Runge-Kutta Nystrom method as starter, see Okuonghae and Ozobokeme (2024).

Example 1: Consider the non-linear problem given by

$$y'' = x(y')^2, \quad y(0) = 1, \quad y'(0) = \frac{1}{2},$$

with an exact solution

$$y(x) = 1 + \frac{1}{2} In\left(\frac{2+x}{2-x}\right).$$

This problem was solved by Adeyefa (2021) with order $p > 3$. The accuracy of the proposed method is measured using maximum absolute error: $|y(x_n) - y_n|$. See Table 1 below.

Table 1: *The exact solution, numerical solution of the new method, and error in the new method for Example 1 with h = 0.003125 and p = 3.*

X	Exact Solution $y(x_n)$	Numerical solution of the new method y_n	Maximum Absolute Error
0.1	1.053175062421496	1.053175062441957	2.04609E-11
0.2	1.103492920586838	1.103492920625291	3.84532E-11
0.3	1.154338911580413	1.154338911634263	5.38502E-11
0.4	1.205989894564679	1.205989894632067	6.73880E-11
0.5	1.258748937687529	1.258748937767193	7.96638E-11
0.6	1.312957225169341	1.312957225260525	9.11837E-11
0.7	1.369009467368560	1.369009467470969	1.02408E-10
0.8	1.427374729847812	1.427374729961602	1.13790E-10
0.9	1.488625715831921	1.488625715957738	1.25816E-10
1.0	1.553481533942765	1.553481534081842	1.39076E-10

Table 1 shows the accuracy of the proposed method; however, it was not possible to compare the results of this method with other existing methods because recently developed methods in the literature have order $p > 3$.

Example 2: Consider the system of ODEs,

$$y_1'' = \frac{-y_1}{r},\ y_1(0) = 1,\ y_1'(0) = 0,$$
$$y_2'' = \frac{-y_2}{r},\ y_2(0) = 1,\ y_2'(0) = 0,$$
$$r = \sqrt{{y_1}^2 + {y_2}^2}, \qquad x \in [0,1].$$

This stiff problem in Example 2 was solved by Okuonghae and Ozobokeme (2024). Table 2 shows the accuracy of the new method.

Table 2: *The solution components (Sol. Comp.), exact, numerical solution and the absolute error for Example 2 with h = 0.003125 and p = 3.*

X	Exact Solution	Numerical solution of the new method	Maximum Absolute Error
0.1	9.953126920352170E-1	9.946873279439612E-1	6.25364E-4
0.2	9.794409517155484E-1	9.806839668301461E-1	1.24301E-3
0.3	9.544083252577275E-1	9.562565371015926E-1	1.84821E-3
0.4	9.198395662993789E-1	9.222744736527696E-1	2.43490E-3
0.5	8.760800744531020E-1	8.790773140286808E-1	2.99723E-3
0.6	8.235670800964600E-1	8.270966699692076E-1	3.52958E-3
0.7	7.628252757105762E-1	7.668519148872761E-1	4.02663E-3
0.8	6.944615733263037E-1	6.989449944613964E-1	4.48342E-3
0.9	6.191590404598302E-1	6.240544121934203E-1	4.89537E-3
1.0	5.376700751278498E-1	5.429284500258345E-1	5.25837E-3

3.2 *Discussion*

The results in Table 2 highlight the accuracy of the new method, but also reveal that numerical accuracy degrades during computation for several reasons. For stiff ordinary differential equations (ODEs), low-order methods can become unstable or inefficient, necessitating techniques with larger stability intervals. Additionally, a constant step size may not be suitable, as a step size that is too large can lead to missed significant events and compounding inaccuracies. Moreover, computations involving floating-point arithmetic can accumulate round-off errors, particularly in iterative methods, where small errors may amplify and propagate, further affecting the results.

4. CONCLUSION

In conclusion, our examination of the proposed method reveals its notable performance advantages when applied to mildly stiff ordinary differential equations (ODEs). The proposed method demonstrates efficiency and accuracy, effectively handling the complexities of this specific class of problems. Its ability to provide reliable solutions with a manageable computational cost makes it a strong candidate for practitioners seeking to tackle mildly stiff ODEs in various applications.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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