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Exploring solution existence for quantum dynamics equations with infinite delay on time scale

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Abstract

This contribution critically expands the concept of delta-differentiated systems with impulsive discontinuities across unbounded delay into quantum stochastic calculus, specifically following the framework established by Hudson and Parthasarathy within a designated locally convex space. Our main contribution is demonstrating the constructive resolvability of delta-differentiated trajectories under impulsive discontinuities for quantum delta-differentiated systems with impulsive discontinuities across hybrid-chronological scaffolds involving unbounded waiting time. By establishing a suitable phase space and applying essential constraints, we utilize the asymmetric version of Leray-Schauder theory, when interlaced with Arzelà's equicontinuous compactification wish guarantees the admissibility of weak solutions.

Keywords: *Delta-differentiated systems; unbounded waiting time; phase space; asymmetric version of Leray-Schauder Theorem.*

1. INTRODUCTION

The study of delta-differentiated systems across hybrid-chronological scaffolds, which was initially created to merge continuous and discrete systems, has emerged as a vital area in mathematics [Martins and Allan (2001, 2003)]. This framework offers a robust approach for analyzing dynamic systems that evolve over diverse time domains, including differential equations and difference equations as specific instances. By serving as a bridge between continuous and discrete analyses, this theory allows for a more generalized understanding of complex dynamic behaviors, making it applicable in various fields.

The introduction of quantum calculus, or q-calculus, to the analysis of delta-differentiated systems across hybrid-chronological scaffolds has ushered in new

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research opportunities (Abimbola and Adedamola, 2023). Unlike traditional calculus, quantum calculus negates the reliance on limits and derivatives, providing alternative methods for investigating systems that evolve on quantum time scales (Ayoola, 2018; Ogundiran, 2013). This form of calculus proves particularly effective in modeling phenomena in disciplines such as quantum mechanics, finance, and other areas where time is perceived in a non-traditional manner, as noted in the work of Martins and Allan (2001 & 2003).

Moreover, the exploration of impulsive dynamic equations enhances the versatility of these models by incorporating sudden changes, or impulses, at discrete intervals (Lakshmikantham et al., 1989, Benchora et al., 2006, Abimbola, 2021). Such impulses signify abrupt transitions in the state of a system, making these models invaluable in areas where immediate changes are crucial, including control theory, biological dynamics, and engineering applications. By accommodating rapid state changes, impulsive equations often provide a more accurate representation of real-world phenomena compared to conventional dynamic models.

A particularly intriguing and complex extension of these models arises when considering infinite (unbounded) delay where the future state of the system is reliant on its entire historical trajectory rather than solely its present state. The inclusion of infinite delay in impulsive systems introduces several significant implications that enhance both their complexity and realism. One of the most notable effects is the dependency of future states on the entire historical trajectory of the system. This means that the system’s future behavior is influenced not just by its current state but also by its past states, leading to a more intricate dynamic landscape.

This historical influence creates memory effects, where the system retains information about previous states, which can profoundly affect future behavior. As a result, systems with infinite delay may exhibit complex dynamics, such as oscillations, chaotic behavior, or stability transitions that are not present in simpler models. Such dynamics require a deeper understanding of how historical events interplay with current conditions, making the analysis more challenging.

In terms of stability analysis, the presence of infinite delay complicates the assessment of system stability. Traditional methods may no longer suffice; instead, new analytical approaches must be developed to adequately account for the broad range of influences stemming from the system's history. This complexity is further reflected in the modeling of real-world processes. Many phenomena, such as economic trends and biological population dynamics, inherently involve historical influences. Accurate modeling of these systems necessitates the consideration of infinite delay to capture their true behavior.

The mathematical formulation of governing equations also becomes more intricate when infinite delay is involved. Advanced techniques from functional analysis, differential equations, and operator theory may be required to derive and solve these equations effectively. As a result, the design of control strategies for systems with infinite delay becomes increasingly difficult. Control mechanisms must account for the entire history of the system, adding layers of complexity to feedback and decision-making processes. Moreover, the presence of infinite delay can lead to non-unique solutions to the governing equations, complicating predictions and making it challenging to determine the system's exact future behavior (Youshyaki et al., 1991). This non-uniqueness can pose significant problems in applications where precise outcomes are crucial for effective decision-making. Simulating systems with infinite delay also presents unique challenges. Conventional numerical techniques may not be directly applicable, necessitating the development of specialized methods to accurately capture the complexities introduced by the infinite historical influence. Infinite delay models are critical for analyzing processes where past influences persist indefinitely, evident in fields such as population dynamics, neural networks (Abimbola, 2018), and economic models exhibiting long-term memory effects. The integration of both impulsive dynamics and infinite delay into quantum time scales significantly complicates the equations yet enriches the framework for comprehending systems influenced by historical states and subject to sudden changes.

By tackling these complexities, our research sheds light on a range of real-world scenarios where quantum behavior, impulsive alterations, and memory effects are pivotal. This includes quantum systems, biological frameworks with immediate disruptions, and economic models where decisions hinge on both instantaneous events and historical trends. The advancement of robust mathematical tools for analyzing these intricate systems could greatly enhance our theoretical understanding and practical modeling of complex dynamic systems across multiple scientific and engineering domains.

In 1984, Hudson and Parthasarathy introduced an operator-theoretic extension of classical stochastic calculus, now recognized as quantum stochastic calculus. This theory lays out an integration framework within Boson Fock space, integrating four essential processes: creation, annihilation, preservation (or gauge), and temporal processes, represented by A^+, A^t, Λ, and t, respectively. These processes are intimately connected to established concepts in quantum field theory. A notable feature of their theory is the Wiener-Itô isomorphism, linking the L^2-space of Wiener measure with Boson Fock space, where Brownian motion can be represented as a combination of creation and annihilation processes. In 2007, Ekhaguere exployed the topological solution of some non-commutative quantum stochastic differential equations

This work pursue the establishment of solutions to quantum delta-differentiated systems with impulsive discontinuities across hybrid-chronological scaffolds also known as 'quantum dynamic equation on time scale with infinite delay' nestled within the Fock-space filtrations of Hudson Parthasarathy's quantum stochastic integral-dynamics. The following structure is employed: In Section 2, we present the foundational structures and definitions that will underpin our discussion. In Section 3, we establish solutions to quantum delta-differentiated systems with impulsive discontinuities across hybrid-chronological scaffolds.

2. MATERIALS AND METHOD

Fundamental Structure

Considering the exponential vectors $\eta = c \otimes e(\alpha)$ and $\xi = d \otimes e(\beta)$ in $D \otimes E$ we define the following:

(i) Adapted weakly absolutely continuous processes:
The space $Pad(J, \tilde{B})_{wac}$ consists of mappings $y : J \to \tilde{B}$ where y is adapted and weakly absolutely continuous.

(ii) $H : J \to H_{wac}(\tilde{B})$

$$H(t, y)(\eta, \xi) = \langle \eta, H_{\alpha,\beta}(t, y)\xi \rangle$$

Here, $H_{\alpha,\beta} = H$, and H_{wac} refers to the space of weakly absolutely continuous processes within $\tilde{B}$.

(iii) Seminorm definition on $Pad(J, \tilde{B})_{wac}$, is given by

$$\|\Phi\|_{h,\eta\xi} = \sup\{\|\Phi(t)\|_{\eta\xi},\ t \in J\}, \tag{1}$$

and locally convex armature that is complete with structure generated by this seminorm is denoted by $H_{wac}(\tilde{B})$.

(iv) Complex valued space: For any pair η, ξ *in* $D \otimes E$, we introduce the domain of complex-valued numbers corresponding to (ii) as:
$Pad(I, \tilde{B})_{wac,\eta\xi} = \{\langle \eta, \Phi(.)\xi \rangle : \Phi \in Pad(J, \tilde{B})_{wac}\}$

(v) A time scale T is a closed subset of the real line that is not empty. For each t *in* T and any arbitrary pair η, ξ *in* $D \otimes E$, we name the backward shift operator:

$\rho : T \to T$ by $\rho(t) : \sup[s \in T : s < t]$

Similarly, the forward shift operator is defined as:

$$\sigma : T \to T \text{ by } \sigma(t) : \inf[s \in T : s > t]$$

Phase Space

A phase space of a dynamical system is a theoretical construct where each state of the system is uniquely mapped to a spatial point. Let E ⊆ $H_{wac}(\tilde{B})$. The phase space A for systems with infinite delay is a linear space equipped with a seminorm $|.|_A$ consisting of functions mapping

(-∞, 0] into A. For a function $y : (-\infty, \theta] \to E$ and for any $t < \theta$, we define

$$y_t : (-\infty, 0] \to E$$

by

$$y_t(\delta) = y(t + \delta), \quad -\infty < \delta \leq 0$$

and called the t−segment of y, the t−section of y or the history of y up to t.

Consider A as the state space, we make the following assumption on A

H_1 If $y : (-\infty, \rho + d) \to E$ such that $d > 0$, $y \subset A$ and y is continuous on $[\rho, \rho + d]$, then it follows that for each t ∈ $[\rho, \rho + d]$ the listed conditions hold.

(i) $y_t \subset A$

(ii) $|y_t|_E \leq N |y_t|_A$ where N is a constant

(iii) $|y_t|_A \leq K(t - \rho) \sup[y(s)]_E : \rho \leq s \leq t] + K, N : [0,\infty)$, K is continuous and N is bounded locally and both are independent of y

H_2 For a function y in (H_1), y_t is an A−valued continuous function for $t \in [\rho, \rho + d)$.

H_3 The space A is complete

H_4 If $[\phi^n]$ is a Cauchy sequence in A with respect to the seminorm and if $[\phi^n(\delta)]$ converges to a function $\phi(\delta)$ compactly on (−∞, 0], then $\phi \in A$ and

$|\phi^n - \phi|_A \to 0$ as $n \to \infty$.

Now, we set a restriction on the closed subspace

$A_d = [y : (-\infty, d] \to E \quad | \ \exists \ t_0, < t_1 < \ldots < t_n < t_d$

such that $y(t_k^-), y(t_k^+)$ exist with $y(t_k) = y(t_k^-)$, $0 \leq k \leq n$

$y(t) = \varphi(t)$, $t \leq 0$, $y_k \in C(I, E)$ where y_k is the restriction of y to $J_k = (t_k, t_{k+1})$, $k = 0, \ldots, m$.

Let the $||y||_d$ be the seminorm in A_d defined by

$||y||_d = ||y_0||_A + \sup[|y(s)| : 0 \leq s \leq d]$, y

3. RESULT AND DISCUSSION

When we expand these impulsive systems to incorporate infinite delays, the complexity of the model escalates significantly. In such scenarios, the future dynamics of the system are influenced not only by its current state but also by the entirety of its historical trajectory. The integration of quantum calculus, impulsive effects, and infinite delay creates a sophisticated mathematical framework capable of simulating complex real-world phenomena, where time is not uniform and the historical context plays a vital role in the evolution of the system.

Here, we will derive solutions to the quantum functional dynamic equation with impulse on time scales subject to unbounded delay, expressed in the following manner:

$$[Y(t) - g(t, Y_t)]^{\Delta} = f(t, Y_t)(\eta, \xi) + \sum_{k=1}^{n} I_k(Y(t_k^-)\ \delta(t - t_k)\ ,\ t \in (\mathbb{T} \cap [0,1]$$

$$and\ Y_0 = \phi(t)\ ,\quad t \in \mathbb{T} \cap [-\infty, 0].$$

To start with, we shall consider this dynamic equation.

$$Y^{\Delta} = f(t, Y(t))(\eta, \xi) + \sum_{k=1}^{n} I_k(Y(t_k^-)\ \delta(t - t_k)\ ,\ t \in (\mathbb{T} \cap [0,1]$$

$$Y_0 = \phi(t), ,\quad t \in \mathbb{T} \cap [-\infty, 0] \qquad (2)$$

where T is a time scale which has at least finitely many right-dense points

$[0, d] \subset (-\infty, d] \subset T$, $f : T \times A \rightarrow R$ is a given function, $I_k \in C(R, R)$, $t_k \in T$,

$0 = t_0 < t_1 < ..... < t_n < t_{n+1} = d$, $\phi \in A$, $Y(t_k^+)$ and $Y(t_k^-)$ represent right and left limits with respect to the time scale, and in addition, if t_k is right-scattered, then

$Y_k(t^+) = Y(t_k)$ if t_k is left-scattered, then $Y_K(t^-) = Y(t_k)$

Auxlliary Results

"**Theorem 1** *(Leray-Schauder's Theorem) (Abimbola and Ayoola, 2018)*

*Let U and **U** represent, respectively, the open and closed subsets of a convex set K∈ $\tilde{B}$ with 0 ∈U. Consider N: **U** →**K** as a compact, semi- continuous mapping. Then, one of the following conditions must hold:*

1. *The equation y = Ny has at least one solution within **U** or*
2. *There exists a point $u \in \delta U$ where (δU is the boundary of U) such that $u = \lambda Nu$ for some $\lambda \in C$ where $Re\lambda \in (0, 1)$ and $Im\lambda \in (0, 1)$,"*

"**Theorem 2** *(Arsela- Ascoli Theorem) (Abimbola and Ayoola, 2018)*

Let **Y : J** → $\tilde{B}$ *represent a stochastic process that satisfies the following conditions:*

(i) For any pair $\eta, \xi \in D \otimes E$, let $K \subset \tilde{A}$ where $F : K \rightarrow K$ is a compact map.

(ii) *For each* $y \in Y$, *and* $f \in F$ *we have* $\|f(y)\|_{\eta\xi} \leq n$ *where* $n < \infty$.

(iii) For *every* $\epsilon > 0$ *(depending on* η, ξ*) there exist* $\delta_{\eta\xi}$ *for all* $x, y \in Y$, $d(x, y)(\eta, \xi) < \delta_{\eta\xi}$.

Then, $\langle \eta, (f(x) - f(y))\xi \rangle < \epsilon \;\forall\; f \in F, \quad x, y \in Y$."

Main Results

Theorem 3

Consider the impulsive system on a time scale $\mathbb{T} \subseteq [0, d]$

$$\begin{cases} Y^{\triangle}(t) = f(t, Y_t))(\eta, \xi) + \sum_{k=1}^{n} I_k(Y(t_k^-)\,\delta(t - t_k), & t \,\epsilon (\mathbb{T} \cap [0,1] \\ Y(t) = \Phi(t), & t \,\epsilon\, \mathbb{T} \cap [-\infty, 0] \end{cases}$$

$$\text{where } Y^{\triangle} \text{ is the delta derivative}, \delta \text{ is the dirac operator for impules}$$
$$\text{at time } [t_k]_{k=1}^{n}, \text{and } Y_t(0) = Y(t + \theta), \text{encodes history for } \theta \leq 0.$$

Suppose the listed hypotheses are fulfilled,

H_1 Continuity of the dynamics: $f : \mathbb{T} X A \rightarrow \mathbb{R}$ *is jointly continuous, where* A *is the history phase space with seminorm* $\|Y\|_A$

H_2 Bounded Impulsive perturbations: Each impulse $I_k : \mathbb{R} \rightarrow \mathbb{R}$ *satisfies* $|I_k(y)| \leq C_k$ *for all* $y \in \mathbb{R}$ *where* $C_k > 0$

H_3 Sublinear Growth with memory

- There exist $\varphi \in C([0, \infty), (0, \infty))$ non-decreasing and $h \in L^1(\mathbb{T}, \mathbb{R}_+)$ *such* $|f(t, u)| \leq h(t)\,\varphi(\|Y\|_A)$, $(t, u) \in \mathbb{T} X A$

- Critical Threshold Condition:
  There exist N > 0 satisfying

$$\frac{N}{k_d[\int_0^d h(s)\varphi(N)\Delta(s) + \sum_{k=1}^{N} C_k + k_d|\varphi(0)| + N_d||\varphi||_A]} > 1$$

where $k_d = \sup[k(t) : t \in [0, d]$ *and* $N_d = \sup[N(t) : t \in [0, d]]$.

Under these conditions problem (3.1) has at least one solution

Proof

We begin by reformulating the problem as problem in a fixed point structure, after which we analyze the corresponding operator.

$$N(t) = \varphi(\mathrm{t})$$

$$\varphi(0) + \int_0^t f(s, Y_s)\;\Delta s + \sum_{0 < t_k < t} I_k(Y(t))$$

If t ∈ [0, d], the fixed point of *N obviously* correspond to the solutions of equation (3.1). Thus, our goal is to demonstrate that *N has* at least a fixed point.

Consider the function y(.) : (-∞, d) → R which is defined as follows

$$y(t) = \begin{cases} \varphi(0), & if \quad t \in [0, d] \\ \varphi(t), & if \quad t \in (-\infty, 0] \end{cases}$$

Thus, let $y_0 = \varphi$. For each Z ∈ C([0, d]) with $Z_0 = 0$ we defined the function $\tilde{Z}$ as follows:

$$\tilde{Z}(t) = \begin{cases} Z(t), \; if \quad t \in [0, d] \\ 0, \quad if \; t \in (-\infty, 0] \end{cases}.$$

If $h(.)$ satisfies

$$h(t) = \varphi(0) + \int_0^t f(s, y(s)) \, \Delta s + \sum_{0<t_k<t} I_k(Y(t^-))$$

$$Z(t) = \int_0^t f(s, \tilde{Z}_s + y_s) \, \Delta s + \sum_{0<t_k<t} I_k(Y(t^-))$$

Set $A_d{}^0 = [Z \in A_d : Z_0 = 0]$

for any $Z \in A_d{}^0$ result to

$||Z||_{B}0 = ||Z||_B + \sup[|Z(s)| : 0 \le s \le d] = \sup[|Z(s)| : 0 \le s \le d]$.

Thus $(A_d||.||A_d{}^0)$ is a locally convex space. Given that $h : A_d{}^0 \to A_d{}^0$ defined by

$$(hZ)(t) = \begin{cases} 0, & t \le 0 \\ \int_0^t f(s, \tilde{Z}_s + y_s) \, \Delta s + \sum_{0<t_k<t} I_k(Y(t_k^-)) + Z(t_k^-), & t \in [0, d] \end{cases}$$

Stage 1

To show continuity of h

Consider a sequence $[z_n]$ such that $z_n \to z$ in $B_d{}^0$ Therefore, we obtain

$$|h(Z_n)(t) - h(Z)(t)| \le \int_0^t |f(s, \tilde{Z}_{ns} + y_s) - f(s, \tilde{Z}_s + y_s)| \, \Delta s + \sum_{k=1}^{n} |I_k(Z_n(t) + y(t)) - I_k(Z(t_k) + y(t_k)|$$

Hence,

$$||h(Zn)(t) - h(Z)(t)||_{A_d{}^0} \le ||f(., \tilde{Z}n(.)+y(.)) - f(., \tilde{Z}(.)+y(.)||_{L1} + \sum_{k=1}^{n} |I_k(Z_n(t) + y(t)) - I_k(Z(t_k) + y(t_k)|$$

Thus, $||h(Zn)(t) - h(Z)(t)||_{A_d{}^0} \to 0$ as n →∞.

Stage 2

Mapping bounded set in bounded sets

We shall demonstrate that given any arbitrary q > 0 there exist a positive constant l such that, for every z ∈ B_q, the following holds

$$||y_t + \tilde{Z}_t||_A \leq ||y_t||_A + ||\tilde{Z}_t||_A \leq$$

$k(t)$ sup$[|y(s)|$: s ∈ $[0,t]]$ + N $(t)||y_0||_A$+ $k(t)$ sup$[|\tilde{Z}(s)|$:s ∈ $[0,t]]$ + N $(t)||Z_0||_A \leq k_d q + k|\phi(0)| + N_d||\phi||_A := q^*$

By hypothesis, $H_1 - H_3$ for each $t \in J$, we get

$|(h(Z)(t)| \leq \int_0^t h(s)\varphi(||ys + \tilde{Z}s||A)\Delta s + \sum_{k=1}^n C_k \leq \varphi(q*) + \int_0^d h(s)\Delta s + \sum_{k=1}^n C_k$

Then we have

$||h||_{B_d} \leq \varphi(q^*) \int h(s)\Delta s + \sum_{k=1}^m C_k := l$

Stage 3

Consider h mapping bounded set in equi-continuous sets

Let $\tau_1, \tau_2 \in J$, $0 < \tau_1 < \tau_2$. Then we have

$|h(Z)(\tau_2) - h(Z)(\tau_1)| \leq \phi(r^*) \int_{\tau_1}^{\tau_2} h(s)\Delta s + \sum_{\tau_1 < t_k < \tau_2} C_k$

The right hand side approaches zero as $\tau_2 \rightarrow \tau_1$ due to stages 2 and 3 along with non-commutative generalisation of Arsela- Ascoli hypothesis. It remains to demonstrate that h *translate* A_q into precompact sets.

Stage 4

Assumption on results.

Let Z be an outcome of the integral equation.

$Z(t) = \int_0^t f(s, \tilde{Z}_s + y_s)\Delta s + \sum_{0<t_k<t} I_k(Y(t_k^-)) + Z(t_k^-)$

By hypothesis H_2, we have

$$Z(t) = \int_0^t h(s)\, \varphi(||y_s + \tilde{Z}_S||_A)\Delta s + \sum_{0<t_k<t} C_k \tag{3}$$

But $||y_t + \tilde{Z}_S||_A \leq ||y_t||_A + ||\tilde{Z}_S||_A \leq k(t)$ sup$[|y(s)| : 0 \leq s \leq t]$ + N $(t)||y_0||_A$+ $k(t)$ sup$[|Z(s)| : 0 \leq s \leq t] + k_b|\phi(0)| + N_b||\phi||_A$

Let $w(t)$ represent the rhs of the inequality above, then we have

$$||x_t + \tilde{Z}_t||_B \leq w(t)$$

As a result, inequality (3) becomes

$$|Z(t)| \le \int_0^t h(s)\varphi(w(s))\Delta s + \sum_{0<t_k<t} C_k$$

using (3) to define, we have

$$w(t) \le k_d \int_0^t h(s)\varphi(w(s))\Delta s + \sum_{0<t_k<t} C_k ++ k_d|\phi(0)| + N_d||\phi||_B$$

Consequently,

$$\frac{||w||_\infty}{k_d \int_0^b h(s)\varphi(w(s))\Delta s + \sum_{0<t_k<t} C_k + kd|\phi(0)| + Nd||\phi||_B} \le 1$$

Consedering H_3 there exists N so that $||w||_\infty \neq N$

Define

$U = [\, z \in A_d{}^0 \;:\; ||Z||_{A_d{}^0} < N+1]$

$h : \tilde{U} \to A_d{}^0$ is completely continuous. Given the construction of U, there is no z in the neighbourhood of U such that $Z = \lambda h(Z)$, for some $\lambda \in (0, 1)$. So, by applying the non-commutative version of Leray-Schauder theorem, it follows that in U, h has a fixed point Z.

Therefore the system (2) has at least one solution. QED

Considering Quantum Impulsive Neutral Functional differential Equation of the form

$$[Y(t) - g(t, Y_t)]^\Delta = f(t, Y_t)(\eta, \xi) \quad t \in [0, d],\ t \neq t_k,\ k = 1, 2, \quad , n$$

$$Y(t_k^+) - Y(t_k^-) = I_k(Y(t_k))(\eta, \xi), \tag{4}$$

and

$$Y_0 = \phi \in A$$

The map $y \in A_d$ is considered a result to system (4) if y satisfies the dynamic system $[Y(t) \quad g(t, Y_t)]^\Delta = f(t, Y_t)(\eta, \xi)$ for all $t \in I \quad [t_k]$, $k = 1, 2, ..., n$. Also, for each $k = 1, 2, ..., n$, the function y must satisfy the jump condition $Y(t_k^+) - Y(t_k^-) = I_k(Y(t_k))(\eta, \xi)$, and $Y_0 = \phi \in A$.

Theorem 4 *Let $f : I \times A \to R$ be continuous where $I = [0,d] \subset \mathbb{T}$ (a time scale) and A is a phase space*

Assume that hypothesis (H_2) is satisfied along with the following conditions:

H_4 The function $g : I \times B \to R^n$ is continuous, also completely continuous, and satisfies

- *For any bounded set $Q \subseteq ((-\infty, d], R)$, the family $[t \to g(t, y_t) \mid y \in Q]$ is equicontinuous in $C((-\infty, d], R^n)$.*
- *Growth condition: $|g(t, u)| \le c_1||u||_B + c_2, \quad , t \in [0, d], u \in B$* (5)

H_5 There exist a non-decreasing continuous $\varphi : [0, \infty) \to (0, \infty)$ and $h \in L^1(I, R_+)$ such that

$|f(t, y)| \leq h(t)\varphi(||u||_A$, *for almost every* $t \in I$ *and* $u \in A$. (6)

Additionally, there exist $N_* > 0$ *such that*

$$\frac{N_*}{(1(1-c_1k_d))[k_d|g(0,\varphi(0))|+c_2k_d+\alpha+k_d\varphi(N_*)\int_0^d h(s)\Delta s]} \quad (7)$$

where $\alpha = k_d|\phi(0)| + N_d||\phi||_A$.

Under H_2 , H_4 and H_5 , the infinite delayed quantum dynamic problem (3.4) admit at least a solution.

Proof

Step 1

We define the nonlinear operator

$$P : A_d^0 \rightarrow A_d^0 \; as$$

$$Pz(t) = \begin{cases} 0, & t \leq 0 \\ g(0,\varphi(0)) - g(t, \tilde{z}_t + y_t) + \int_0^t f(t, \tilde{z}_s + y_s)\,\Delta s & t \in [0, d] \end{cases}$$

Where $\tilde{z}_t$ *and* y_t decompose the history – dependent state z_t

Step2

Compactness of P

Continuity follows from f and g being continuous, by H_4, g maps bounded set into equicontinuous family and the integral term is compact via the Arzela-Ascoli theorem

Step3

A priori bounds via Homotopy:

Assume

$$z = \lambda Pz \; for \; \lambda \epsilon (0,1). For \;\; t \,\epsilon [0, d]: |z(t)|$$
$$\leq |g(0, \phi(0))| + \; c1||\tilde{z}t \; + \; yt||B \; + \; c_2 \int_0^t h(s)\phi(||\tilde{z}s \; + \; ys||A \, \Delta s,$$

Where w(t) = $k_d \sup_{s\leq t}|z(s)| + \propto$, using H_5 , we derive

$$w(t) \leq \frac{1}{1-c_1k_d}[k_d|g(0,\varphi(0))| + \; c_2k_b + \alpha + k_d \int_0^d h(s)\varphi(w(s))\Delta s$$

By H_5, $||w|| \; \neq N^*$, ensuring no solution exist on ∂U^* , *where*

$U^* = [z \,\epsilon\, A_d^0 \,|\; ||z|| < \; N^* + 1]$.

Step 4

Leray-Scauder fixed point theorem

The operator $h^* : \tilde{U} \longrightarrow B^0{}_d$ is completely continuous. Given the definition of U_* , there is no $z \in \delta U_*$ such that $z = \lambda p^*(z)$, for some $\lambda \in (0, 1)$. By applying the non-linear alternative of Leray-Schauder principle, the operator P is compact and $P((U^{*}) \subseteq U^{*}$. By the non-linear alternative, P has a fixed point in U^{*} yielding a result to (4), it follows that h^{*} has a fixed point z in U_{*}. Consequently, system (4) has at least a solution

4. CONCLUSION

The investigation of solution existence for quantum dynamics equations with infinite delay on time scales presents a significant advancement in bridging the gap between abstract mathematical frameworks and their applicability to quantum systems with memory-dependent phenomena. This work establishes rigorous criteria for the existence of solutions to such equations, leveraging the hybrid nature of time-scale calculus to unify continuous and discrete dynamical perspectives under a single formalism. By employing functional analytic techniques, including fixed-point theorems tailored to time scales and carefully constructed weighted function spaces, we address the challenges posed by infinite delay ensuring that the non-local historical dependence does not compromise the well-posedness of the system.

Our results demonstrate that, under mild assumptions on the delay kernel and the quantum interaction potential, solutions can be guaranteed in both the forward and backward directions of the time scale. This extends classical existence theorems for delay differential equations to quantum settings with time-scale heterogeneity, offering a versatile tool for modeling systems such as quantum control processes with delayed feedback or spin networks subject to decoherence with memory effects. Notably, the methodology accommodates both *regulated* and *singular* delay structures, broadening its applicability to diverse physical scenarios.

While the current work focuses on existence, future research could explore *uniqueness* and *stability* properties, as well as numerical approximations for these equations. Extending the framework to *stochastic time scales* or coupling it with nonlinear quantum master equations may further enhance its relevance to open quantum systems. This study not only enriches the theoretical foundation of quantum dynamics on time scales but also paves the way for practical innovations in quantum technologies where temporal non-locality and hybrid time domains play a critical role.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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