BENIN JOURNAL OF PHYSICAL SCIENCES
BJPS Vol. 2(1), June, pg. 122-128 (2025)
ISSN 3043-6931(Print)
3043-694X(Online)
www.bjps.org.ng

Generalizations of Some Basic Theorems of Real Analysis

K. E. OSAWARU[1,*] and R. N. NWAKA[2]

[1]*Department of Mathematics, University of Benin, Benin City, Nigeria*
[2]*Department of Mathematics and Statistics, University of Delta, Agbor, Nigeria*

Received: 18/04/2025 ***Accepted:*** 25/06/2025

Abstract

We define for each $\alpha \in [0,1]$, the nested $\alpha-$level sets for fuzzy sets and the nested $\alpha-$intervals for fuzzy numbers as a generalization of the nested intervals of the Real line ($\mathbb{R}$). Furthermore, we prove two essential theorems: the nested $\alpha-$level sets theorems and the intermediate-value theorem for fuzzy numbers. The results generalize existing results in the literature.

Keywords: *Nested interval, Fuzzy set, Fuzzy number, $\alpha-$level sets, Real number.*

1. INTRODUCTION

Some of the basic results in Real analysis are the theorems of nested intervals, the intermediate value theorem, Bolzano Weistress theorem amongst others. See Apostle (1974). These are crucial results as they form the background for the study of deferential and integral calculus. The calculus being defined on the Real line is restrictive since it is crisp set based. The crisp set is rigid in the sense that the membership value of elements is either 1 or 0. It's by nature precise and thus with-it ambiguity, imprecision which are characteristic of real-world situations and complex systems can't be modelled sufficiently. But a more general definition called fuzzy set introduced by Zadeh (1965) which has recently received a lot of attention, is an efficient tool for describing imprecision. Fuzzy set theory has tremendous power for application in modeling against the crisp set which is conceptually precise. Thus, there are numerous extensions of the concept in other research areas. One of such is the consideration of fuzzy-valued functions for fuzzy differentiation and integration. The Hukuhara differentiability (H-differentiability) of the fuzzy-valued functions arrest such restrictions. It consists of increasing the diameter of solutions level sets as time increases; an improvement

*Corresponding author e-mail: **kelly.osawaru@uniben.edu**

on traditional differentiability. An improvement of the H-differentiability is presented in Armand et al. (2018) as they introduced the strong and weak generalized differentiability. Here, derivative exists and the solution of a fuzzy differential equation may have decreased length of the support, but the uniqueness is lost. However, this disadvantage is seen as an advantage since it is possible to choose the singular points where the support of the solution changes its monotonicity. This helps us to obtain reversible solutions, stable and almost periodic solutions and asymptotic behavior of solutions to the fuzzy differential equations(See Stefanini and Bede, 2009). The approach followed by Stefanini (2010) generalized Hukuhara differentiability (gH-derivative) introduced in Stefanini and Bede (2009), which is based on a generalization of the H-difference between two intervals. In Bede and Gal (2005), the important theorems in fuzzy arithmetic, such as a fuzzy intermediate value theorem, Bolzano's theorem, mean value theorem for integral and mean value theorem for gH-derivative and Rolles Theorem were stated and proved for fuzzy-valued functions.

In this work, we define, characterize and prove some results that will pave way for the consideration of differentials of Real-valued function defined on fuzzy sets. This will allow for the generalization of function defined on a collection of sets (closed) and crisp set in the literature. This approach is significant as the emphasis of our proposed fuzzy differential depends on an ambient space not necessarily on the function value.

2. RESULTS AND DISCUSSION

The results of this study are presented as follows:

1: Section [2.1]: We define and characterize the concept of nested α −level sets for fuzzy sets and nested α −interval for each $\alpha \in [0,1]$ for fuzzy numbers. We show that they generalize the nested intervals of the Real line. Further we state and prove the Nested Theorem for fuzzy sets and show results it generalizes.

2: Section [2.2] : We state and prove the Intermediate value Theorem for continuous function defined on a fuzzy set. We substantiate our claims of generality by stating and proving some corollaries.

Nested $\boldsymbol{\alpha}$ −level sets and the Nested $\boldsymbol{\alpha}$ −level sets Theorem

We recall the following results from set theory:

For any two non-empty sets A and B.

(i) $A \triangle B = A\backslash B \cup B\backslash A$

(ii) If $A \subset B$ then $A \triangle B = B\backslash A$

(iii) If $A = B$ then $A \triangle B = \emptyset$

The first definition and remarks show a generation of nested intervals of the Real line. Here, instead of having intervals of the Real line, we have intervals defined by level sets of a fuzzy number, a generalization of intervals of the Real line.

Definition 1 (Enclosed level sets of a fuzzy set): Let A be a fuzzy set of X (i.e there is $\mu: X \to [0,1]$ such that $\mu(x) = 0$ if $x \notin A$ and $\mu(x) \in (0,1]$ if $x \in A$). Let $A_\alpha = \{x \in X: \mu(x) \geq \alpha \}$ be an α −level set of A for any fixed $\alpha \in (0,1]$.

Define $[\mathrm{A}]_\gamma = \{A_\gamma: A_\alpha \subseteq A_\gamma \subseteq A_\beta \quad \alpha, \gamma, \beta \in [0,1]\}$ (1)

Then we call $[\mathrm{A}]_\gamma$ an enclosed level set of A for any α and β. Note that $\alpha \leq \gamma \leq \beta$.

Definition 2 (Nested level sets): Let A be a fuzzy set of X , A_α an α −level set of A for any $\alpha \in [0,1]$ and $[\mathrm{A}]_\gamma$a collection of level sets of A for any $\alpha, \beta, \gamma \in [0,1]$. Then the sequence

$[A]_i = \{A_\gamma: A_{\alpha_i} \subseteq A_\gamma \subseteq A_{\beta_i} \quad \alpha_i, \beta_i, \gamma \in [0,1]\}$ $i = 1,2, \ldots.$ (2)

such that $[A]_{i+1} \subset [A]_i$ for each i, is called a sequence of nested level sets of fuzzy set A.

Remarks 3:

(i) Suppose in Definition 1, $X = \mathbb{R}$. Then A is a fuzzy number of $\mathbb{R}$ if $u : \mathbb{R} \to [0, 1]$ is a membership function of $\mathbb{R}$ satisfying the following properties:

(a) u is upper semi-continuous on $\mathbb{R}$,

(b) u is fuzzy convex (i.e C={ $x \in \mathbb{R}$ such that : $u(x) \in [0,1]$} is a convex set)

(c) u is normal (i.e there exist $x \in \mathbb{R}$ such that : $u(x) = 1$), and

(d) $cl\{x \in \mathbb{R} : u(x) > 0\}$ is compact with respect to the usual topology on $\mathbb{R}$, where cl denotes the closure of a subset.

Thus,

$A_\alpha = \{x \in \mathbb{R}: \mu(x) \geq \alpha, \ \alpha \in (0,1]\}$ is a closed interval.

So, the enclosed level set of Equation (1) is an enclosed closed interval and $[A]_i$ $i = 1,2, \ldots$of Equation (2) is a sequence of level sets of fuzzy number A . If in addition we have that $[A]_{i+1} \subset [A]_i$, then $[A]_i$is a sequence of nested level sets of fuzzy number A.

(ii) Suppose in (i) of Remarks 3, A is a crisp Real number (i. e. $u : \mathbb{R} \to \{0, 1\}$), then

$[A]_i = \{x \in A : \mathrm{a_i} \leq x \leq \mathrm{b_i} \}$ $i = 1,2, \ldots$ (3)

is a sequence of closed intervals. If in addition we have that $[A]_{i+1} \subset [A]_i$, then $[A]_i$is a sequence of nested intervals.

Example: Consider a fuzzy set A with the membership function u: $[-10,10] \to [0,1]$ defined as $u(x) = e^{(-0.02(x-1)^2)}$. Then A is a fuzzy set of "Real numbers in the interval $[-10,10]$ close to 1". Thus, membership values of the Real numbers increase as x gets closer to 1. Also, A is a fuzzy number as $[-10,10] \subset \mathbb{R}$ and u satisfies (i) (a) - (d) of Remarks 2.

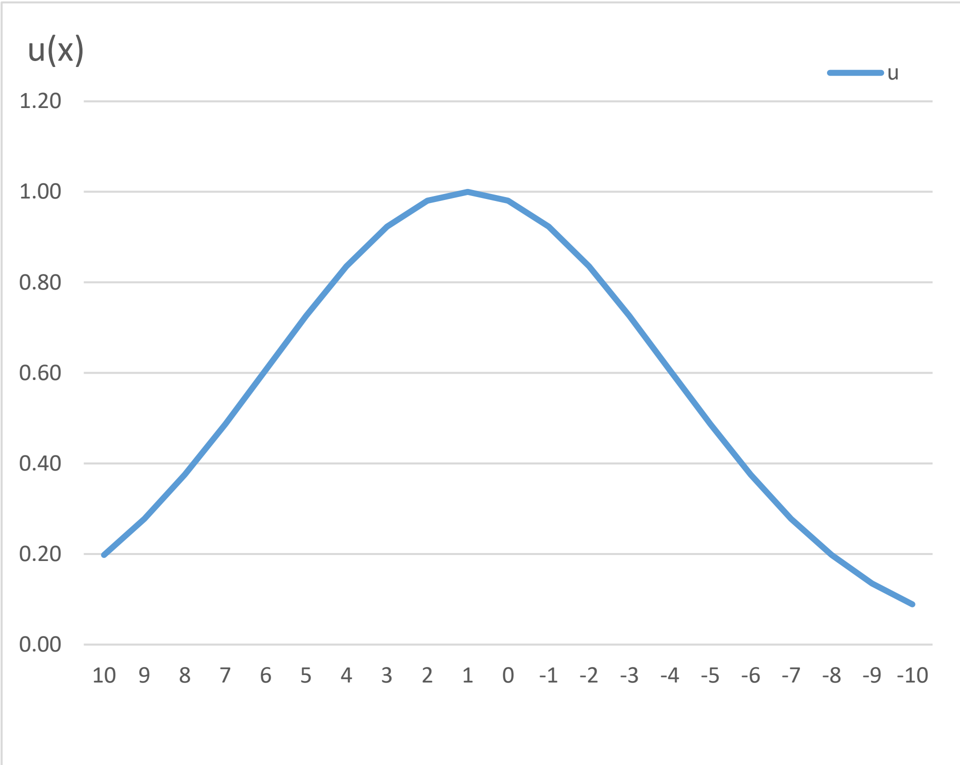

Figure 1: *Graph of membership function* u*:* $[-10,10] \to [0,1]$ *such that* $u(x) = e^{(-0.02(x-1)^2)}$

Let $\alpha = 0.5$ then $A_{0.5} = [-4,7]$. If $\alpha = 0.8, \gamma = 0.6$ *and* $\beta = 0.4$. then $[A]_{0.6} = A_{0.6} = [-4,6]$ is an enclosed 0.6 −level set of A. If $\alpha_1 = 0.8, \alpha_2 = 0.7, \alpha_3 = 0.6$, $\beta_1 = 0.3, \beta_2 = 0.4, \beta_3 = 0.5$. Then $[A]_1 = \{A_\gamma : A_{0.8} \subseteq A_\gamma \subseteq A_{0.3}\}$, $[A]_2 = \{A_\gamma : A_{0.7} \subseteq A_\gamma \subseteq A_{0.4}\}$ and $[A]_3 = \{A_\gamma : A_{0.6} \subseteq A_\gamma \subseteq A_{0.5}\}$. Clearly, $[A]_3 \subseteq [A]_2 \subseteq [A]_1$, thus $[A]_i$ is a sequence of nested level sets of fuzzy set A for i = 1,2 and 3.

In the sequel, denote a fuzzy set A as a fuzzy set of X (any non-empty set) with the membership function $\mu: X \to [0,1]$ and a fuzzy number A as a fuzzy set of $\mathbb{R}$ (the Real line) or a subset of $\mathbb{R}$ with the membership function $\mu: \mathbb{R} \to [0,1]$ satisfying (i) (a) - (d) of Remarks 2.

Theorem 4: Let $[A]_i$ be a nested level sets of the fuzzy set A for each $i = 1,2, \dots$. If $lim_{\{i\to\infty\}}(A_{\beta_i} \triangle A_{\alpha_i}) = \emptyset$, then there exist a unique A_{γ_0} such that $A_{\gamma_0} \in [A]_i$ for every i.

Proof: By Equation 1.2, $A_{\alpha_i} \subset A_{\alpha_{i+1}}$ and $A_{\beta_{i+1}} \subset A_{\beta_i}$. Since $A_{\alpha_i} \subset A_{\beta_i}$ for every i, the sequence $\{A_{\alpha_i}\}$ is nondecreasing and bounded above by A_{β_1}. Similarly, the sequence $\{A_{\beta_i}\}$ is nonincreasing and bounded below by A_{α_1}. Thus there are A_{γ_0} and A'_{γ_0} such that $A_{\alpha_i} \to A_{\gamma_0}, A_{\alpha_i} \subseteq A_{\gamma_0}$ and that $A_{\beta_i} \to A_{\gamma_0}, A_{\gamma_0} \subseteq A_{\beta_i}$ as $i \to \infty$. Using the fact that $A_{\beta_i} \backslash A_{\alpha_i} \to \emptyset$ as $i \to \infty$, we have that $A_{\gamma_0} = A'_{\gamma_0}$ and $A_{\alpha_i} \subseteq A_{\gamma_0} \subseteq A_{\beta_i}$ for every i. Thus $A_{\gamma_0} \in$ for every i.

Suppose A_{γ_0} is not unique, then there is $A_{\gamma_1} \in [A]_i$ also. Define $\epsilon = A_{\gamma_1} \triangle A_{\gamma_0}$. Now since $A_{\alpha_i} \to A_{\gamma_0}$ and $A_{\beta_i} \to A_{\gamma_0}$ whenever there is an integer N_1 such that $A_{\alpha_{N_i}} \supset A_{\gamma_0} \backslash \epsilon$; there is an integer N_2 such that $A_{\beta_{N_2}} \subset A_{\gamma_0} \cup \epsilon$. These set relations imply that $[A]_i$ cannot contain A_{γ_1} for each i beyond N_1 and N_2 . Then A_{γ_1} is not in every $[A]_i$. The proof is complete.

Corollary 5: Let $[A]_i$ be a nested intervals of level sets of the fuzzy number A for each $i = 1,2, \ldots$. If $lim_{\{i \to \infty\}}\left(A_{\beta_i} - A_{\alpha_i}\right) = 0$ (where$\left(A_{\beta_i} - A_{\alpha_i}\right) = inf_{\ a \in A_{\beta_i}, b \in A_{\alpha_i}} \mid a - b \mid$), then there exist a unique A_{γ_0} such that $A_{\gamma_0} \in [A]_i$ for every i.

Proof: The proof follows from the proof of Theorem 4 as a fuzzy set is a generalization of the fuzzy number.

Corollary 6: Let $[A]_i$ be a nested intervals of intervals of the Real line for each $i = 1,2, \ldots$. If $lim_{\{i \to \infty\}}(B_i - \mathrm{A_i}) = 0$ (where$(B_i - \mathrm{A_i}) = inf_{\ a \in \mathrm{A_i}, b \in \mathrm{B_i}} \mid b - a \mid$), then there exist a unique A_0 such that $A_0 \in [A]_i$ for every i.

Proof: The proof follows from the proof of Theorem 4 as a fuzzy number is a generalization of the Real number.

Corollary 7 (Theorem of Nested Intervals of the Real line): Let $[A]_i$ be a nested intervals of the Real number A for each $i = 1,2, \ldots$. If $lim_{\{i \to \infty\}}(\mathrm{b_i} - \mathrm{a_i}) = 0$, then there exist a unique x_o such that $x_o \in [A]_i$ for every i.

Proof: The proof follows from the proof of Theorem 4 as a fuzzy number is a generalization of the Real number.

Next, we prove the intermediate value theorem for fuzzy sets. The result is significant because it is fuzzy set based. Thus, paving the way for the foundation of a fuzzy differential calculus of functions defined on fuzzy sets.

Intermediate-value Theorem for level sets

Theorem 8: Let $[A] = \{A_\gamma : A_\alpha \subseteq A_\gamma \subseteq A_\beta \ \alpha, \beta, \gamma \in [0,1]\}$ be a collection of closed sets of a fuzzy set A of X and let $f : [A] \to \mathbb{R}$ be continuous on $[A]$, $A_{\gamma_i} \in [A]$ for each i and $A_{\alpha_i} \to A_{\alpha_0}$. Then $A_{\alpha_0} \in [A]$ and $f\left(A_{\alpha_i}\right) \to f\left(A_{\alpha_0}\right)$.

Proof: Since for each i, $A_{\gamma_i} \supseteq A_\alpha$ by the theorem of nested sequence of sets. Similarly, $A_{\alpha_0} \subseteq A_\beta$ so $A_{\alpha_0} \in [A]$. If $A_\alpha \subseteq A_{\alpha_0} \subseteq A_\beta$, then $f\left(A_{\alpha_i}\right) \to f\left(A_{\alpha_0}\right)$. Same holds if $A_{\alpha_0} = A_\alpha$ or A_β . The proof is complete.

Theorem 9 (Intermediate-value Theorem for a Fuzzy Number): Let $[A] = \{A_\gamma : A_\alpha \leq A_\gamma \leq A_\beta \ \alpha, \beta, \gamma \in [0,1]\}$ be a collection of closed set A be a fuzzy number and $f: [A] \to \mathbb{R}$ continuous on $[A]$, $c \in \mathbb{R}, f(A_\alpha) < c$ and $f(A_\beta) > c$. Then there is at least one A_{α_0} on $[A]$ such that $f(A_{\alpha_0}) = c$.

Proof: Put $A_\alpha^1 = A_\alpha$ and $A_\beta^1 = A_\beta$. Then observe that $f\left(\frac{(A_\alpha^1 \cup A_\beta^1)}{2}\right) = c$, $f\left(\frac{(A_\alpha^1 \cup A_\beta^1)}{2}\right) \geq c$ or less than $f\left(\frac{(A_\alpha^1 \cup A_\beta^1)}{2}\right) < c$. If $f\left(\frac{(A_\alpha^1 \cup A_\beta^1)}{2}\right) = c$, choose $A_{\alpha_0} = \frac{(A_\alpha^1 \cup A_\beta^1)}{2}$ if it equals c and the result is shown. If $f\left(\frac{(A_\alpha^1 \cup A_\beta^1)}{2}\right) > c$, then put $A_\alpha^2 = A_\alpha^1$ and $A_\beta^2 = \frac{(A_\alpha^1 \cup A_\beta^1)}{2}$. If $f\left(\frac{(A_\alpha^1 \cup A_\beta^1)}{2}\right) < c$, then put $A_\alpha^2 = \frac{(A_\alpha^1 \cup A_\beta^1)}{2}$ and $A_\beta^2 = A_\beta^1$. In each case, we have $f(A_\alpha^1) < c$ and $f(A_\beta^1) > c$. Next, if $c = f\left(\frac{(A_\alpha^2 \cup A_\beta^2)}{2}\right)$ the result is shown. If $f\left(\frac{(A_\alpha^2 \cup A_\beta^2)}{2}\right) < c$, we put $A_\alpha^3 = \frac{(A_\alpha^2 \cup A_\beta^2)}{2}$ and $A_\beta^3 = A_\beta^2$.

Continuing, we find A_{α_0} on $[A]$ in a finite number of steps or we find a sequence $[A]_i$ of closed intervals of closed intervals each of which each one of the two halves of the proceeding one, and for which we have $A_{\beta_i} - A_{\alpha_i} = (A_\beta^1 - A_\alpha^1)/2^{i-1}$, $f(A_{\alpha_i}) < c, f(A_\beta^1) > c$ for each i. From Corollary 4 , there is a unique interval in all these intervals and $\lim\limits_{i\to\infty} A_{\alpha_i} = A_{\alpha_0}$ and $\lim\limits_{i\to\infty} A_{\beta_i} = A_{\alpha_0}$. By Theorem 7 , we conclude that $f(A_{\alpha_i}) \to f(A_{\alpha_0})$ and $f(A_{\beta_i}) \to f(A_{\alpha_0})$. From $A_{\beta_i} - A_{\alpha_i} = (A_\beta^1 - A_\alpha^1)/2^{i-1}$, $f(A_{\alpha_i}) < c$, $f(A_\beta^1) > c$ for each i, it follows that $f(A_{\alpha_0}) \leq c$ and $f(A_{\alpha_0}) \geq c$. This implies that $f(A_{\alpha_0}) = c$. The proof is complete.

Corollary 10 (Intermediate-value Theorem for Interval of the Real Number): Let $[A] = \{A_0 : A \leq A_0 \leq B\}$ be a collection of closed intervals the Real number and $f: [A] \to \mathbb{R}$ continuous on $[A]$, $c \in \mathbb{R}, f(A) < c$ and $f(B) > c$. Then there is at least one A_0 on $[A]$ such that $f(A_0) = c$.

Proof: The proof follows from the proof of Theorem 9 as the $\alpha -$ level sets of a fuzzy number generalize the interval of the Real line.

Corollary 11 (Intermediate-value Theorem for the Real Number): Let $[A] = \{x : a \leq x \leq b\}$ a closed interval of the Real line and $f: [A] \to \mathbb{R}$ continuous on $[A]$, $c \in \mathbb{R}, f(a) < c$ and $f(b) > c$. Then there is at least one x_0 on $[A]$ such that $f(x_0) = c$.

Proof: The proof follows from the proof of Theorem 9 as the fuzzy number generalizes the Real number.

3. CONCLUSION

In this paper, we have defined, characterized and proved some results that are useful in the study of a differential of Real-valued function defined on a fuzzy set rather than the fuzzy-valued function in the literature. Our results proved to be more natural as the idea of imprecision that exist in a system is actually noticed on the ambient space and not necessarily on the function. The results are shown to generalize existing results.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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