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Adams-Type Formulae for the treatment of Initial Value Problems (IVPs) in Stiff Stochastic Ordinary Differential Equations (SODEs)

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Abstract

Stochastic Ordinary Differential Equations (SODEs) are often insoluble via the use of analytic techniques due to randomness that characterize SODEs. Stiffness in SODEs introduces complexity in that numerical method for approximating the solution of stiff SODEs are required to be A-stable, a stringent condition that is attainable by implicit method only. This article presents a class of A-stable methods, that derived via the use of Ito Taylor expansion, Taylor series expansion and method of undetermined coefficients. Boundary locus plot method is used to carry out the stability analysis of this class of methods. The methods derived herein are shown to be A-stable for step number $k \leq 12$ and are of order p=1. Numerical test on two standard linear and nonliear stiff SODEs problems in the literature are carried out. Solutions generated using the proposed class of methods show that the numerical method mimic the analytic solution. The class of method derived in this article are well suited in treatment of stiff SODEs and compete favorable with existing methods.

Keywords: A-stable; Mean Square Stable; Pathwise error; Order

1. INTRODUCTION

Real life phenomena are often laced with uncertainties or randomness. Mathematical Modeling of these phenomena would require tracking the randomness they exhibit. Some real-life phenomena whose mathematical model exhibit fluctuations or random noise include: Chemical reactions, finance, Epidemiological models, etc. The models formulated using initial value problem of the form:

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$$dX(t) = f\big(X(t)\big)\,dt, \quad X(t_0) = X_0, \quad t \in [t_o, T], \quad X \in \mathbb{R}^m, \tag{1.1}$$

are not suitable where randomness exists. To account for randomness that exist, Mathematical models that are formulated using Stochastic Ordinary Differential Equations (SODEs) of the form:

$$\begin{gathered} dX(t) = f(X(t))dt + g(X(t))dW(t), \quad X(t_0) = X_0, \quad t \in [t_o, T], \quad X \in \mathbb{R}^m, \\ f: [t_0, T] \times \mathbb{R}^m \to \mathbb{R}^m, \quad g: \mathbb{R}^d \to \mathbb{R}^{d\times m}, \end{gathered} \tag{1.2}$$

are preferred over deterministic (1.2) Brugnano et al (2000).

In (1.2), $f\big(X(t)\big)$ is the drift coefficient, $g\big(X(t)\big)$ is the diffusion coefficient (the stochastic term) and $W(t)$ is the independent Wiener process, with increment $\Delta W(t) = W(t + \mathrm{h}) - W(t)$ is a Guassian random variable $N(0, \mathrm{h})$. h is the step size.

The presence of stochastic term in (1.2), contribute to the complexity of finding the exact solution via analytic means, Samsudin (2017). Hence, the need for efficient numerical scheme that would yield approximate solution to (1.2) is desired.

The SODE (1.2) is said to possess unique solution, if the functions $f\big(X(t)\big)$ and $g\big(X(t)\big)$ satisfy the Lipschitz and linear growth bound conditions, Kloeden and Platen (1992). That is, for a constant L,

$$\| f(X) - f(Y) \|^2 \leq L^2 \| X - Y \|^2, \tag{1.3a}$$

$$\| g(x) - g(y) \|^2 \leq L^2 \| x - y \|^2. \tag{1.3b}$$

Also, if for a constant C,

$$\| f(x) \|^2 + \| g(x) \|^2 \leq C(1 + \| x \|^2). \tag{1.4}$$

Expectedly, numerical methods design for numerical approximation of (1.2) that satisfy conditions (1.3) to (1.4), must contain part that would track the trajectory of the solution path of stochastic trajectory. The development of numerical methods for integrating SODEs (1.2) is still enjoying research attention as evident in the literature, Brugnano et al (2000), Samsudin et al (2017). Numerical methods for integrating SODEs (1.2) are classified into explicit, semi-implicit and implicit methods, Samsudin et al (2017). The numerical scheme for SODEs (1.2) is said to be explicit, if it is explicit in both deterministic and stochastic components. Numerical method is said to Implicit, if it implicit in any or both of the deterministic and stochastic components. Majority of the existing methods are explicit as remarked in Samsudin et al (2017). The stiffness in SODEs arises from differences in time scales or highly non-linear dynamics, Toccino and Senosiain (2014). Numerical methods that can efficiently handle Stiff SODEs are preferred to be A-Stable. A-stable methods possess stable regions which includes the entire left half of the complex plane and are found amongst implicit methods, Hairer and Wanner (1996), Butcher (2008). Implicit methods have better convergence and wider stability region compared to explicit and semi-implicit methods, Tian and

Burrage (2001). This article is concerned with the construction of a sub-class of second derivative Linear Multistep methods known as Adams Type formula, this is of particular property this sub class of method possesses. That is, they are known to be zero-stable a priori, Hairer and Wanner (1996).

This article is arranged as follows: Section 2 is on the construction of proposed class of method, the order and stability properties of the methods are presented in section 3. Section 4 is on numerical experiments, while section 5 is the conclusion.

2. MATERIALS AND METHOD

2.1 Construction of Method

Second Derivative Adams type method for integrating SODE (1.2) of the form:

$$X_{n+1} = X_n + h\sum_{j=0}^{k}\beta_j f_{n+j-k+1} + \sum_{j=0}^{k}\gamma_j g_{n+j-k+1}\Delta W_n + \frac{1}{2}\sum_{j=0}^{k}\eta_j g'_{n+j-k+1}g_{n+j-k+1}[(\Delta W_n)^2 - h] \quad (2.1)$$

is introduced. The method (2.1), is implicit in both deterministic and stochastic components.

The terms $f_{n+j-k+1}$ and $g_{n+j-k+1}$in (1.2) are the drift and the diffusion components. $g'_{n+j-k+1}$ is the derivative of g with respect to t, ΔW_n is the Guassian random variable, $W_n \sim N(0,\mathrm{h})$ and h is the step size.

Parameters: $\beta_j, \gamma_j, \eta_j, j = 0,1,\dots,k$ are to be determined, with conditions;

$$\sum_{j=0}^{k-1}\beta_j = 1, \qquad \sum_{j=0}^{k-1}\gamma_j = 1, \qquad \sum_{j=0}^{k-1}\eta_j = 1. \quad (2.2)$$

Conditions in (2.2), ensures that the method (2.1) is consistent, Platen and Kloeden (1992).

The Linear difference operator $L(X(t_n), h)$, associated with (2.1) is given by:

$$\begin{aligned} L(X(t_n), h) \coloneqq X(t_n + h) - X(t_n) - h\sum_{j=0}^{k}\beta_j\, f\left(X(t_n + h + \tau_j)\right) \\ -\Delta W(t_n + h)\sum_{j=0}^{k}\gamma_j\, g\left(X(t_n + h + \tau_j)\right) \\ -1/2\,[(\Delta W(t_n + h))^2 - h]\sum_{j=0}^{k}\eta_j\,(L_1 g)\left(X(t_n + h + \tau_j)\right). \end{aligned} \quad (2.3)$$

The parameters of (2.1) are obtained as follows:

Firstly, Taylor expands the deterministic part of (2.3) about (t_n, X_n), such that

$$X(t_n + h) = X(t_n) + hf_n + h^2\!/\!_2\, G_n + h^3\!/\!_6\, H_n + h^4\!/\!_{24} J_n + O(h^5), \quad (2.4)$$

$$f\left(X(t_n + h + \tau_j)\right) = f_n + (j)h\, G_n + (jh)^2/2\, H_n + (jh)^3/6\, J_n + O(h^4). \quad (2.5)$$

Substituting (2.4) and (2.5) into (2.3), collect powers of h, and match independent derivatives. This yields the deterministic moment (consistency) conditions:

$$\beta_0 + \beta_1 = 1, \qquad -\beta_0 = -1/2. \tag{2.6}$$

Secondly, for the stochastic terms, Ito Taylor expand (2.3) about (t_n, X_n) to obtain

$$g\left(X(t_n + h + \tau_j)\right) = g(t_n) + \tau_j (L_0 g)(t_n) + O(h^2), \tag{2.7}$$

$$(L_1 g)\left(X(t_n + h + \tau_j)\right) = \left(L_1 g(t_n)\right) + O(h). \tag{2.8}$$

L_0 and L_1 are as defined in Kloeden and Platen (1992).

The Linear operator of the stochastic part is:

$$L(X(t_n); h) = \Delta W\left(\sum_{j=0}^{k} \gamma_j\right) g(t_n) + 1/2\,((\Delta W)^2 - h)\left(\sum_{j=0}^{k} \eta_j\right)\left(L_1 g(t_n)\right) + O(h^{3/2}). \tag{2.9}$$

Matching Ito–Taylor yields:

$$\sum_{j=0}^{k} \gamma_j = 1, \qquad \sum_{j=0}^{k} \eta_j = 1.$$

For fully implicit choices concentrated at $j = k$ this condition holds automatically, Kloeden and Platen (1992). (since $j - k = 0$ for $j = k$).

$$\gamma_j = \begin{cases} 0, & j < k, \\ 1, & j = k, \end{cases} \;. \tag{2.10a}$$

$$\eta_j = \begin{cases} 0, & j < k, \\ 1, & j = k. \end{cases} \tag{2.10b}$$

For step number $k = 1$,

The moment equation $m = 0{,}1$ from (2.6) gives: $\beta_0 = 1/2$, $\beta_1 = 1/2$,

With $\gamma_0 = 0,\ \gamma_1 = 1,\ \eta_0 = 0,\ \eta_1 = 1$, the resulting method (2.1) for k=1 is

$$X_{n+1} = X_n + h\left(1/2\,(f_n + f_{n+1})\right) + g_{n+1}\,\Delta W_n + 1/2\, g'_{n+1} g_{n+1}((\Delta W_n)^2 - h). \tag{2.11}$$

For step number $k = 2$;

The moment equations $m = 0, 1, 2$ are

$$\begin{aligned} \beta_0 + \beta_1 + \beta_2 &= 1, \\ -2\beta_0 - \beta_1 &= -1/2\,, \\ 4\beta_0 + \beta_1 &= 1/3\,. \end{aligned}$$

Solving the arising system of linear equation, yields:

$$\beta_0 = -1/12\,, \quad \beta_1 = 2/3\,, \quad \beta_2 = 5/12$$

With $\gamma_0 = \gamma_1 = 0$, $\gamma_2 = 1$, $\eta_0 = \eta_1 = 0$, $\eta_2 = 1$, the scheme becomes:

$$X_{n+1} = X_n + h(-1/12\, f_{n-1} + 2/3\, f_n + 5/12\, f_{n+1}) + g_{n+1}\, \Delta W_n + 1/2\, g'_{n+1} g_{n+1}((\Delta W_n)^2 - h) \quad (2.12)$$

For $k = 3$,

$$\beta_0 = 1/24\,,\ \beta_1 = -5/24\,,\ \beta_2 = 19/24\,,\ \beta_3 = 3/8,$$

Diffusion coefficients

$$\gamma_0 = \gamma_1 = \gamma_2 = 0, \quad \gamma_3 = 1,$$

$\eta_0 = \eta_1 = \eta_2 = 0, \quad \eta_3 = 1$.

The scheme is:

$$X_{n+1} = X_n + h\left(\frac{1}{24} f_{n-2} - 5/24\, f_{n-1} + 19/24\, f_n + 3/8\, f_{n+1}\right) + g_{n+1}\, \Delta W_n + 1/2\, g'_{n+1} g_{n+1}[(\Delta W_n)^2 - h]. \quad (2.13)$$

For k=4

$$X_{n+1} = X_n + h(-19/720\, f_{n-3} + 53/360\, f_{n-2} - 11/30\, f_{n-1} + 323/360\, f_n + 251/720\, f_{n+1}) + g_{n+1}\, \Delta W_n + 1/2\ g'_{n+1} g_{n+1}((\Delta W_n)^2 - h), \quad (2.14)$$

For k=5

$$X_{n+1} = X_n + h(3/160\, f_{n-4} - 173/1440\, f_{n-3} + 241/720\, f_{n-2} - 133/240\, f_{n-1} + 1427/1440\, f_n + 95/288\, f_{n+1}) + g_{n+1}\, \Delta W_n + 1/2\ g'_{n+1} g_{n+1}((\Delta W_n)^2 - h). \quad (2.15)$$

3. RESULT AND DISCUSSION

3.1 Order and Stability Analysis

Proposition 1.

Scheme (2.1) is of strong order 1.

Proof:

Assume $f, g \in C^3(\mathbb{R}^n)$. Also assume f, g and satisfy Lipschitz and linear growth condition,

Then the one step map of scheme (2.1) is given by:

$$X_{n+1} = X_n + h \sum_{j=0}^{k} \beta_j\ f(X_{n+j}) g' + \sum_{j=0}^{k} \gamma_j\ g(X_{n+j})\, \Delta W_n + 1/2 \sum_{j=0}^{k} \eta_j\ (X_{n+j})((\Delta W_n)^2 - h) \quad (3.1)$$

The Ito expansion of the exact solution of (1.2) is:

$$X(t_{n+1}) = X(t_n) + f_n h + g_n \Delta W_n + 1/2\,(f_n f_n' + 1/2\, g_n^2 f_n'')h^2 + (g_n f_n')I_{(1,0)} + (f_n g_n' + 1/2\, g_n^2 g_n'')I_{(0,1)} + (g_n g_n')\,1/2\,((\Delta W_n)^2 - h) + R_{n+1}, \quad (3.2)$$

where;

$$\begin{aligned} I_{(0)} &= h, \\ I_{(1)} &= \Delta W_n, \\ I_{(1,1)} &= \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} d\,W_u\, dW_s = 1/2\,((\Delta W_n)^2 - h), \\ I_{(1,0)} &= \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} d\,W_u\, ds, \\ I_{(0,1)} &= \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} d\,u\, dW_s. \end{aligned} \quad (3.3)$$

The expectation of Ito integrals satisfies the following relation:

$$\boldsymbol{E}\left|\boldsymbol{I}_{(\mathbf{1,0})}\right|^{\mathbf{2}} = \boldsymbol{O}(\boldsymbol{h}^{\mathbf{3}}), \qquad \boldsymbol{E}\left|\boldsymbol{I}_{(\mathbf{0,1})}\right|^{\mathbf{2}} = \boldsymbol{O}(\boldsymbol{h}^{\mathbf{3}}), \quad (3.4)$$

The local truncation error (lte), is obtained by equating (3.1) and (3.2) as:

$$\begin{aligned} \tau_{n+1} = {} & \left(f_n - \textstyle\sum_{j=0}^{k} \beta_j\, f_{n+j}\right)h + \left(g_n - \textstyle\sum_{j=0}^{k} \gamma_j\, g_{n+j}\right)\Delta W_n \\ & + \left(1/2\,(g_n g_n') - 1/2 \textstyle\sum_{j=0}^{k} \eta_j\, g_{n+j}'\right)((\Delta W_n)^2 - h) \\ & + (g_n f_n')\, I_{(1,0)} + (L^0 g_n)\, I_{(0,1)} + \tilde{R}_{n+k}. \end{aligned} \quad (3.5)$$

$$\begin{aligned} \tau_{n+1} = {} & \left(1 - \textstyle\sum_{j=0}^{k-1} \beta_j\right) f_n h + \left(1 - \textstyle\sum_{j=0}^{k-1} \gamma_j\right) g_n \Delta W_n \\ & + \left((1 - \textstyle\sum_{j=0}^{k-1} \eta_j\right)1/2\, g_n g_n'((\Delta W_n)^2 - h) \\ & + (g_n f_n')\, I_{(1,0)} + (L^0 g_n)\, I_{(0,1)} + \tilde{R}_{n+k}, \end{aligned} \quad (3.6)$$

applying the consistency conditions (2.2), (3.6) reduces to:

$$\tau_{n+1} = (g_n f_n')\, I_{(1,0)} + (L^0 g_n)\, I_{(0,1)} + \tilde{R}_{n+k}. \quad (3.7)$$

Taking the norm of both sides of (3.6) and squaring the resultant gives:

$$\| \tau_{n+1} \|^2 = \| (g_n f_n')\, I_{(1,0)} + (L^0 g_n)\, I_{(0,1)} + \tilde{R}_{n+k} \|^2. \quad (3.8)$$

$$\| \tau_{n+1} \|^2 \leq \| (g_n f_n')\, I_{(1,0)} + (L^0 g_n)\, I_{(0,1)} \|^2 + \| \tilde{R}_{n+k} \|^2. \quad (3.9)$$

Taking expectation of both sides of (3.7) to obtain:

$$\mathbb{E}\, \| \tau_{n+1} \|^2 \leq \mathbb{E}\, \| (1/2\,(L^0 f_n) - \theta_k G_{n+1})h^2 \|^2 + \mathbb{E}\, \| (g_n f_n')\, I_{(1,0)} + (L^0 g_n)\, I_{(0,1)} \|^2 + \mathbb{E}\, \| \tilde{R}_{n+k} \|^2. \quad (3.10)$$

Recall (3.4), then inequality (3.10) becomes:

$$\mathbb{E}\, \| \tau_{n+1} \|^2 \leq C h^3. \quad (3.11)$$

The global error e_{n+1} incurred by approximating the solution (1.2) using (2.1) is obtained as:

$$e_{n+1} = \tau_{n+1} + X(t_n) - X_n. \tag{3.12}$$

Taking expectation of both sides of (3.12) yield:

$$\mathbb{E}|e_{n+1}|^2 \leq (1 + Ch)\,\mathbb{E}|e_n|^2 + C\,\mathbb{E}|\tau_{n+1}|^2. \tag{3.13}$$

Substituting (3.11) into (3.13) gives:

$$\mathbb{E}|e_n|^2 \leq (1 + Ch)\,\mathbb{E}|e_{n-1}|^2 + Ch^3. \tag{3.14}$$

By discrete Grönwall's lemma (Platen and Kloeden (1992)), uniformly for $t_n \leq T$,

$$\mathbb{E}|e_n|^2 \leq Ch^2.$$

Therefore, (2.1) is of strong order 1 and the proof is established.

3.2 Mean Square Stability Analysis

We study mean square stability using the linear test equation (1.5), with parameters $\lambda \in C$ and $\mu \in R$. Applying (1.5) to (2.1) yields:

$$\mathrm{X}_{\mathrm{n+k}} = X_{n+k-1} + h\lambda\sum_{j=0}^{k}\beta_j X_{n+j} + \mu\Delta W_n\sum_{j=0}^{k}\gamma_j X_{n+j} + \frac{1}{2}\mu^2[(\Delta W_n)^2 - h]\sum_{j=0}^{k}\eta_j X_{n+j} \tag{3.15}$$

Simplifying (3.15) to obtain:

$$\mathrm{X}_{\mathrm{n+k}}\left[1 - h\lambda\beta_k - \mu\gamma_k\Delta W_n - \frac{1}{2}\mu^2\eta_k\left((\Delta W_n)^2 - h\right)\right] = X_{n+k-1} + h\lambda\sum_{j=0}^{k-1}\beta_j X_{n+j} + \mu\Delta W_n\sum_{j=0}^{k-1}\gamma_j X_{n+j} + \frac{1}{2}\mu^2[(\Delta W_n)^2 - h]\sum_{j=0}^{k-1}\eta_j X_{n+j} \tag{3.16}$$

Let ρ be the shift operator, that is

$$X_{n+j} = \rho^j X_n, \quad j = 0,1,\dots,k.$$

Then (3.16) becomes,

$$\rho^{\mathrm{k}}\mathrm{X}_{\mathrm{n}}\left[1 - h\lambda\beta_k - \mu\gamma_k\Delta W_n - \frac{1}{2}\mu^2\eta_k\left((\Delta W_n)^2 - h\right)\right] = \rho^{k-1}X_n + h\lambda\sum_{j=0}^{k-1}\beta_j\rho^j X_n + \mu\Delta W_n\sum_{j=0}^{k-1}\gamma_j X_n + \frac{1}{2}\mu^2[(\Delta W_n)^2 - h]\sum_{j=0}^{k-1}\eta_j\rho^j X_n \tag{3.17}$$

Factorizing X_n, and let $\xi = \Delta W_n$.

$$\rho^{\mathrm{k}}\mathrm{X}_{\mathrm{n}}\left[1 - h\lambda\beta_k - \mu\gamma_k\Delta W_n - \frac{1}{2}\mu^2\eta_k\left(\xi^2 - h\right)\right] = \left[\rho^{k-1} + h\lambda\sum_{j=0}^{k-1}\beta_j\rho^j + \mu\Delta W_n\sum_{j=0}^{k-1}\gamma_j + \frac{1}{2}\mu^2[(\Delta W_n)^2 - h]\sum_{j=0}^{k-1}\eta_j\rho^j]\right]X_n \tag{3.18}$$

$$\rho^{\mathrm{k}}\left[1 - h\lambda\beta_k - \mu\gamma_k\Delta W_n - \frac{1}{2}\mu^2\eta_k\left(\xi^2 - h\right)\right] = \left[\rho^{k-1} + h\lambda\sum_{j=0}^{k-1}\beta_j\rho^j + \mu\Delta W_n\sum_{j=0}^{k-1}\gamma_j + \frac{1}{2}\mu^2[(\Delta W_n)^2 - h]\sum_{j=0}^{k-1}\eta_j\rho^j]\right] \tag{3.19}$$

Stability Polynomial associated with scheme (2.1) is given as:

$$P(\rho;\xi) = \rho^k a_k(\xi) - \sum_{j=0}^{k-1} b_j(\xi)\rho^j = 0, \tag{3.20}$$

where

$$a_k(\xi) = 1 - h\lambda\beta_k - \mu\gamma_k\,\xi - 1/2\,\mu^2\eta_k(\xi^2 - h),$$
$$b_j(\xi) = h\lambda\beta_j + \mu\xi\gamma_j + 1/2\,\mu^2(\xi^2 - h)\eta_j, \quad 0 \le j \le k-2, \tag{3.21}$$
$$b_{k-1}(\xi) = 1 + h\lambda\beta_{k-1} + \mu\xi\gamma_{k-1} + 1/2\,\mu^2(\xi^2 - h)\eta_{k-1}.$$

For each fixed realization of ξ, the method is absolute, stable if all roots of $P(\rho;\xi) = 0$ satisfy $|\rho(\xi)| < 1$. For $k = 1$, the equation reduces to

$$\rho\, a_1(\xi) = b_0(\xi), \tag{3.22}$$

so the *stability function* is

$$P(\rho;\xi) = \frac{b_0(\xi)}{a_1(\xi)}, \tag{3.23}$$

with

$$a_1(\xi) = 1 - h\lambda\beta_1 - \mu\gamma_1\xi - 1/2\,\mu^2\eta_1(\xi^2 - h),$$
$$b_0(\xi) = 1 + h\lambda\beta_0 + \mu\gamma_0\xi + 1/2\,\mu^2\eta_0(\xi^2 - h). \tag{3.24}$$

$$\mathbb{E}[\,P(\rho;\xi)|^2\,] < 1. \tag{3.25}$$

This expectation can be computed using the Gaussian moments: Platen and Kloeden (1992).

$$\mathbb{E}[\xi] = 0, \quad \mathbb{E}[\xi^2] = h, \quad \mathbb{E}[\xi^4] = 3h^2.$$

Hence, (2.1) is mean-square stable whenever

$$\mathbb{E}[\,|R(\xi)|^2\,] < 1.$$

3.3 A- Stability analysis

Using Boundary locus plot, the stability plots for k=2 to 10 are given in figure 1, showing the scheme is A- stable.

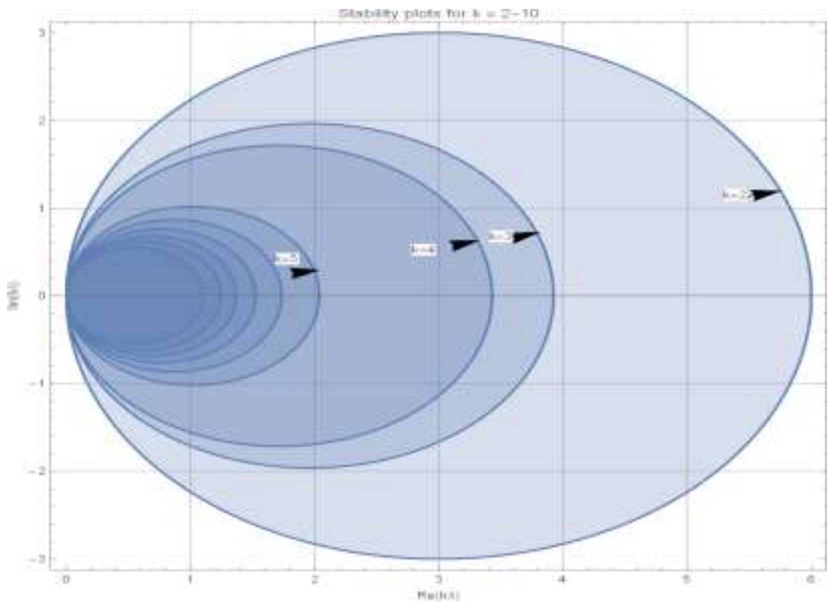

Figure 1: *Stability plot of method (2.1) for k=2-10*

In Figure 1, the stability regions of (2.1) are shown for step numbers k=2,3,4,…,10. The regions of absolute stability for each step number k, shows that the class of Adams Type method (2.1) is A stable for k=2,3,4,…,10.

3.4 Numerical Experiments

Numerical experiments on benchmark stiff SODEs to illustrate the efficiency of the proposed methods (2.1), which is reference Adams Type Formula (ATF) herein compare with the order p=1 method Semi-Implicit Taylor Schemes for Stiff Rough Differential Equations (SITS), developed in Riedel and Wu (2020). The numerical solution generated using ATF are also compared with the exact solution of the two problems considered. The pathwise Error (PE) obtained using CMSSDE and SITS are compared with ATF.

Definition 1: Pathwise Error (PE), (Milstein and Tretyakov (2004))

The **pathwise error** refers to the difference between the numerical and exact solution along individual sample paths.

For each realization of Brownian motion (each path), the error is:

$$\mathrm{PE} = \sup_{0\le n\le N} |X(t_n,\omega) - X_n(\omega)|.$$

The proposed methods are tested on the following problems:

Test Problem 1: Linear Stiff SODE, Yin and Gan (2015)

Consider the linear one-dimensional stiff SODE with multiplicative noise:

$$dX(t) = -20X(t)\,dt + 5X(t)\,dW(t), \quad 0 < t \le T, \quad X(0) = 1,$$

with parameters $\lambda = -20$, $\mu = 5$. The exact solution is given by:

$$X(t) = \exp\left[\left(-20 - \frac{1}{2}\cdot 25\right)t + 5W(t)\right] = \exp[-32.5t + 5W(t)].$$

Test Problem 2: Non-Linear stiff SODE, Reidel and Wu (2020)

Consider the non-linear S0DE with additive noise: (Stochastic Nagumo Equation)

$$dx(t) = \big(x(t) - x^3(t)\big)dt + dW(t), \quad t \in (0,1], \quad x(0) = -3.0$$

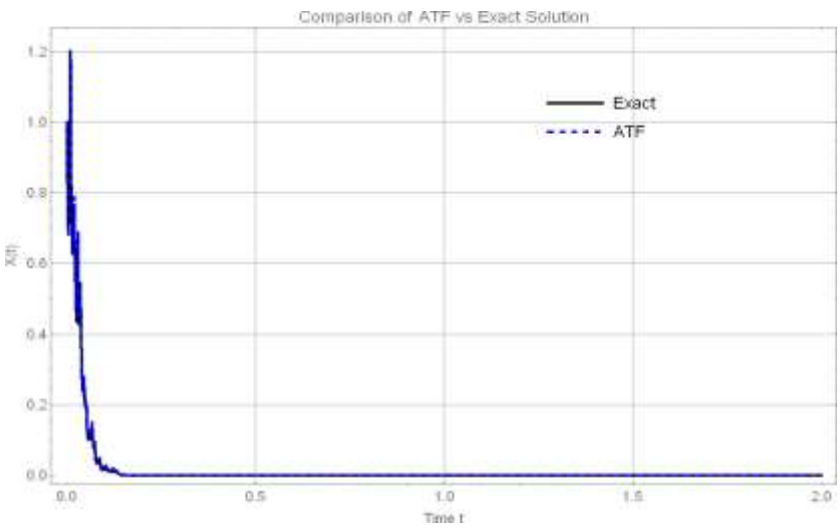

Figure 2: *Solution to test problem 1*

Table 1: *Comparison of pathwise error (PE) for the test problem 2*

H	ATF	SITS
0.007813	2.8100E-2	2.9995E-2
0.003906	1.6050E-2	1.6201E-2
0.001953	8.3000E-3	8.3910E-3
0.000977	4.0100E-3	4.0810E-3

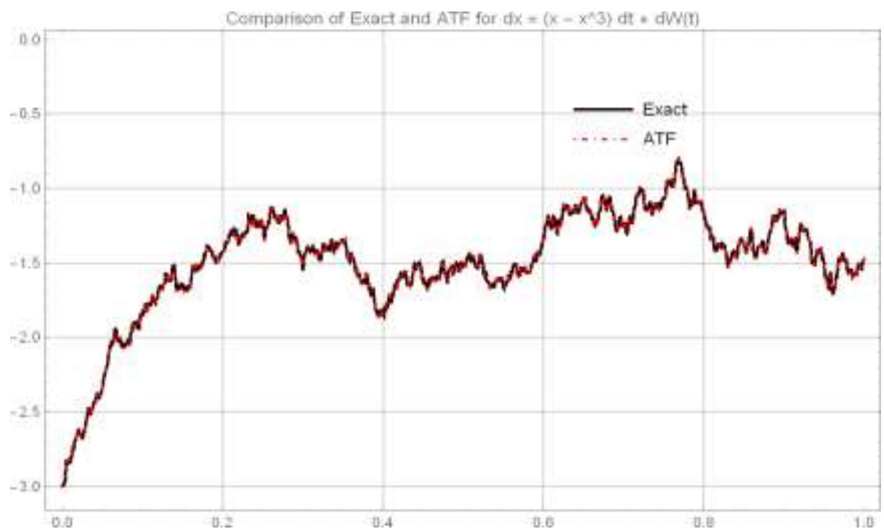

Figure 3: *Solution to test problem 2*

The solution plot of problem 1 shows that the solution generated by ATF track the trajectory of exact solution of problem 1 as shown in Fig. 2. In Table 2, Pathwise error was computed for both ATF and SITS and were compared. The result show that ATF is more accurate than SITS of Reidel and Wu (2020). The solution plots to test problem 2 is shown in Fig. 3, the proposed method track the random motion of the problem 2.

4. CONCLUSION

In this paper, a new class of second derivative Adams Type method for SODEs is developed. The method show promise as the inclusion of stochastic terms makes it efficient in approximating solution to SODEs. The A-stability property of method proposed makes suitable for stiff SODEs. The theoretical analysis and numerical experiments collectively demonstrate that the scheme possesses desirable stability and convergence properties, which are essential for the accurate and efficient simulation of stiff stochastic systems. The effectiveness of the proposed class of methods is demonstrated by the application on two standard test problems. Which show that the method is suitable for integrating stiff SODEs.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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